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INGELS

On the Generalized Frequency

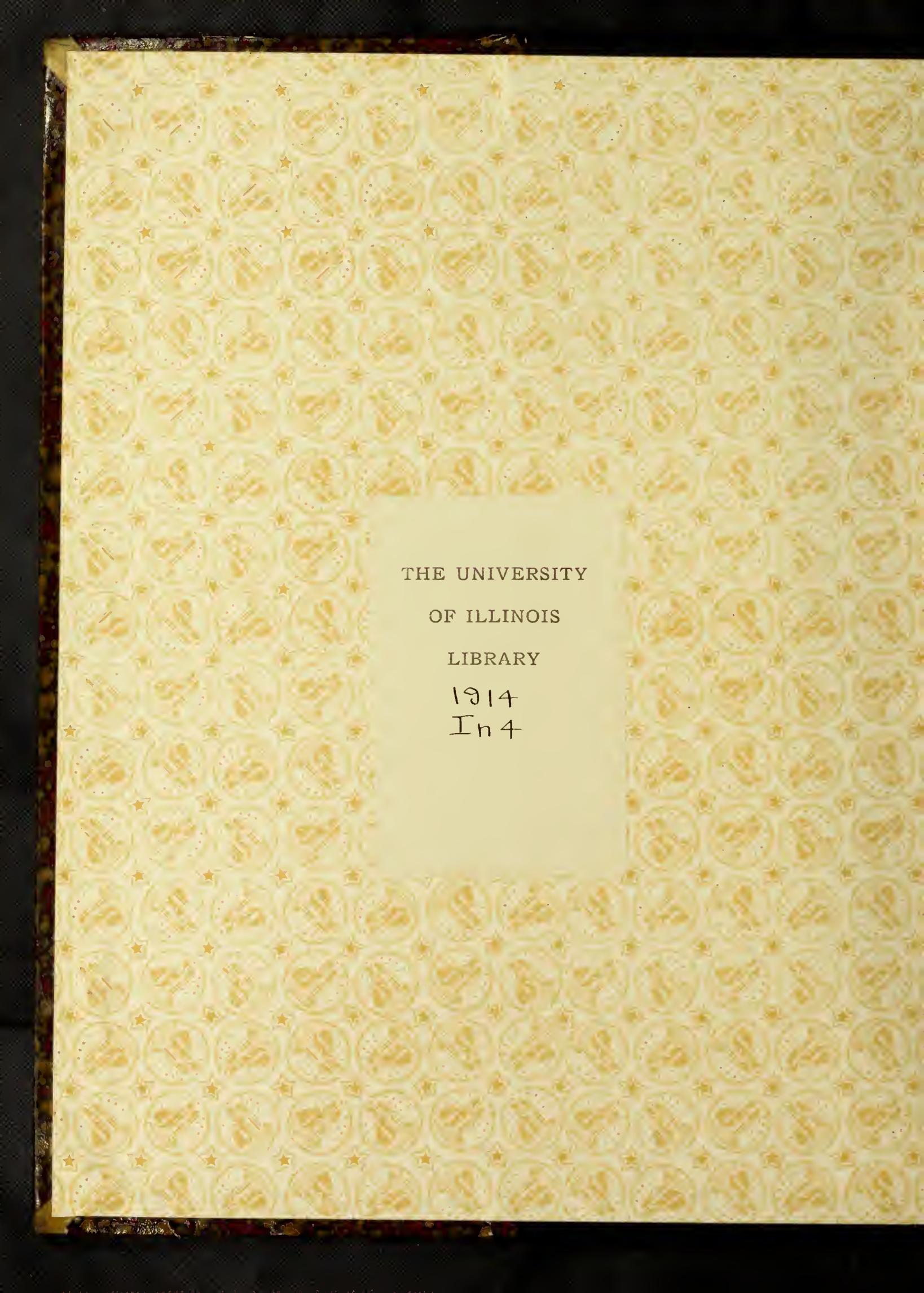
Functions of Edgeworth

Mathematics

A. M.

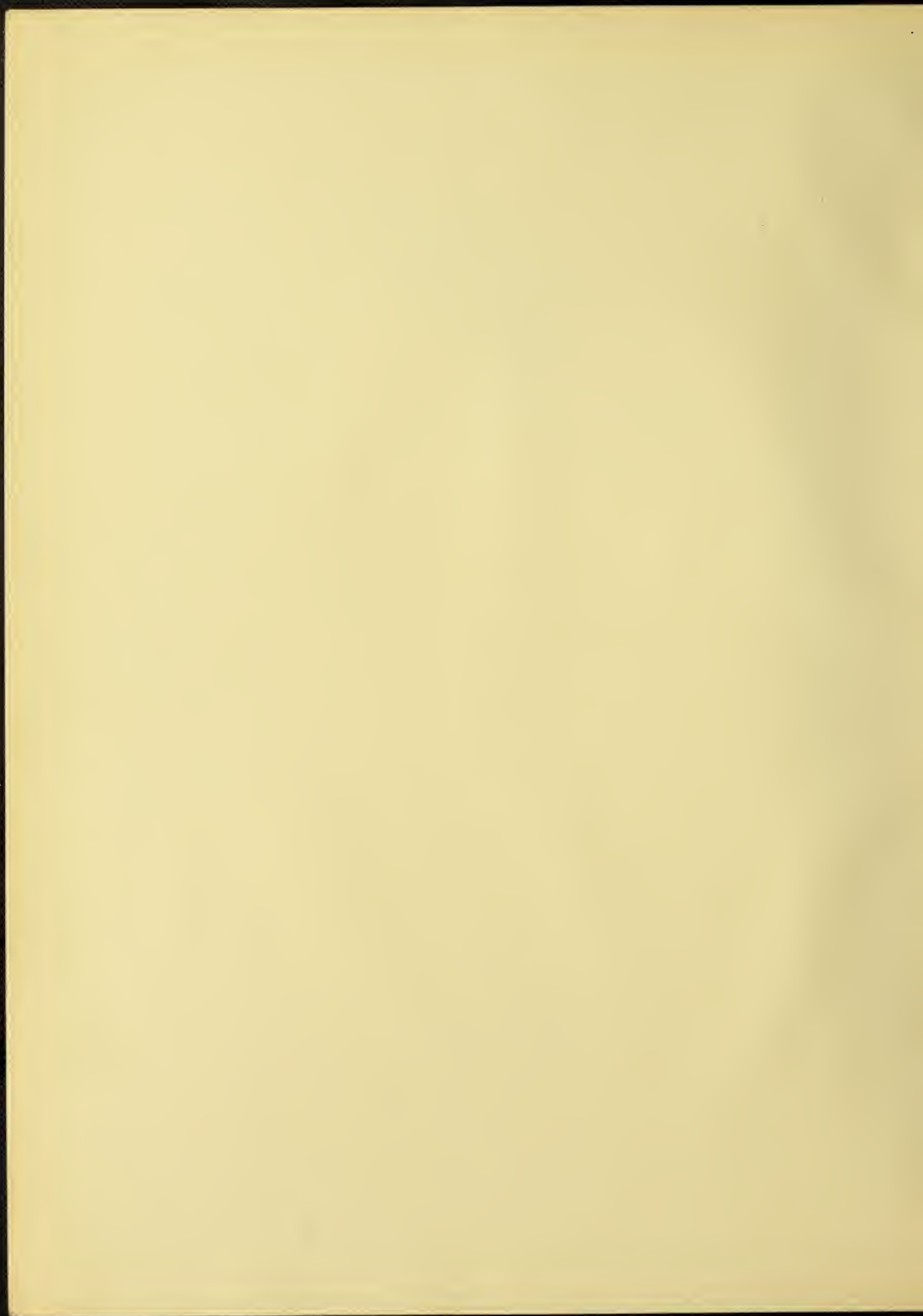
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ON THE GENERALIZED FREQUENCY FUNCTIONS
OF EDGEWORTH

BY

NELLE LOUISE INGELS
Ph.B. Greenville College, 1911

THESIS

Submitted in Partial Fulfillment of the Requirements for the

Degree of

MASTER OF ARTS

IN MATHEMATICS

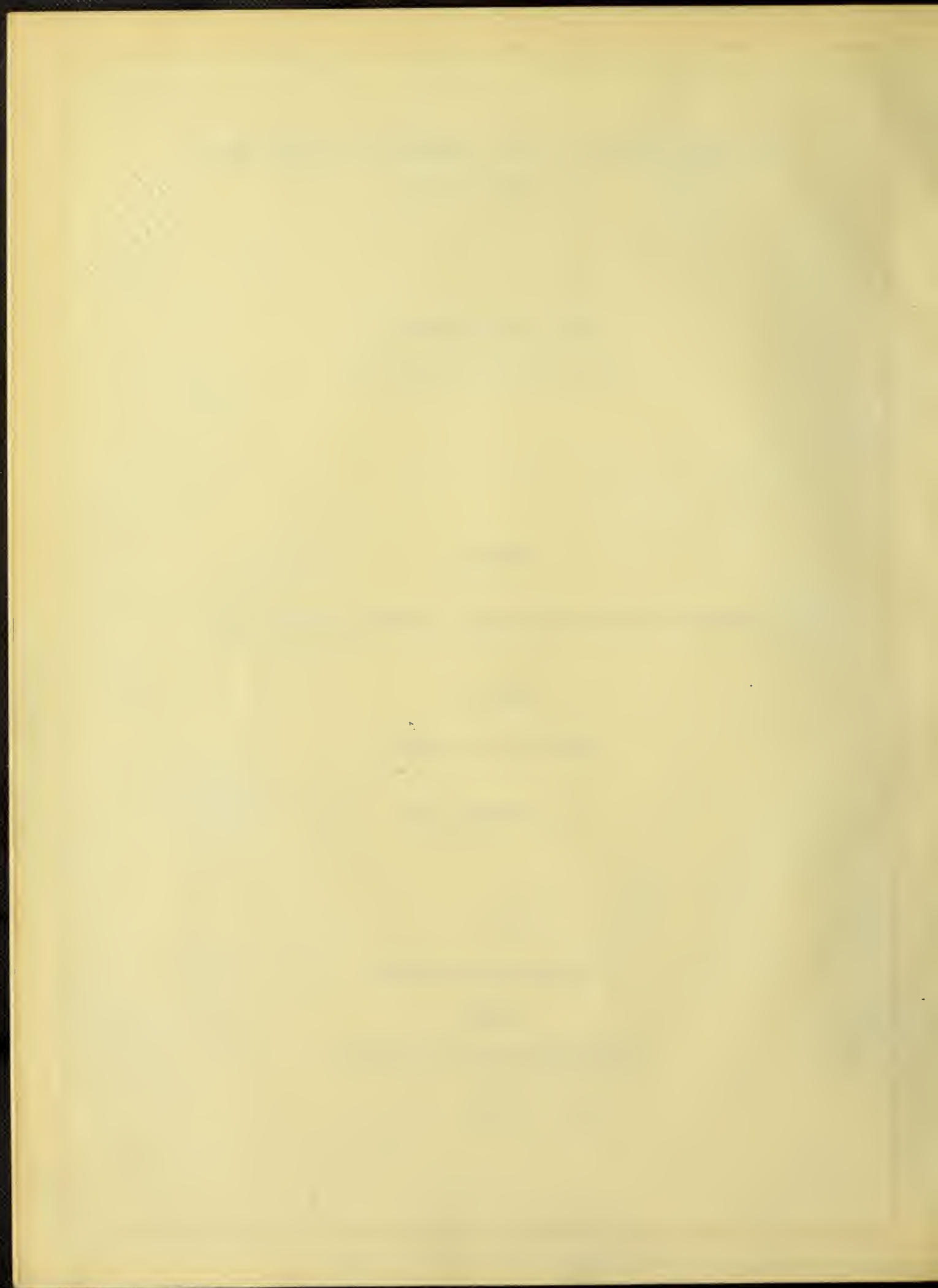
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1914



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UNIVERSITY OF ILLINOIS
THE GRADUATE SCHOOL

May 30, 1914

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Nelle Louise Ingels

ENTITLED *On the Generalized Frequency
Functions of Edgeworth*

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE

DEGREE OF *Master of Arts*

H. L. Rich

In Charge of Major Work

J. Townsend

Head of Department

Recommendation concurred in:

Committee

on

Final Examination

284590



TABLE OF CONTENTS.

Section.	Page.
I. Introduction	1.
II. Transformation $X = x^2$	3.
III. Transformation $X = x^3$	11.
IV. Transformation $X = a(\xi + \kappa \xi^2)$	13.
V. Transformation $X = a(\xi + \kappa \xi^2 + \lambda \xi^3)$	18.
Conclusion	22.



ON THE GENERALIZED FREQUENCY FUNCTIONS OF EDGEWORTH.

I. INTRODUCTION.

While many distributions follow the normal law, if allowance is made for deviations due to random sampling, there are well known classes of variates which do not follow this law. The distribution of statures of certain classes of men fits the normal curve. It is only reasonable to expect that if linear measurements follow the normal law, the corresponding similar surfaces and volumes should be distributed in accord with some transformation of that law.

Let $x_1, x_2, x_3, \dots, x_n$ be variates of a distribution where

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}}$$

If $X_1, X_2, X_3, \dots, X_n$ are a new system of variates where

$$X_1 = kx_1, \quad X_2 = kx_2, \quad \dots, \quad X_n = kx_n,$$

the type of the curve is not changed, as this transformation alters only the scale or the size of the modulus.

If, however, a function, other than a linear function of the variates is substituted for each variate, the form of the curve will be changed. For example, if we make the transformations

$$X_1 = kx_1^2, \quad X_2 = kx_2^2, \quad X_3 = kx_3^2, \quad \dots, \quad X_n = kx_n^2,$$

and
$$X_1 = kx_1^3, \quad X_2 = kx_2^3, \quad X_3 = kx_3^3, \quad \dots, \quad X_n = kx_n^3,$$

the transformed frequency distributions may be regarded as distributions of surfaces and volumes. It is but natural to assume that certain distributions are in the nature of similar surfaces and volumes whose linear dimensions follow the normal law. In more general terms, it is but natural

to expect that certain observed variates are functions of more fundamental elements where those fundamental elements are normally distributed.

This conception of transformation of variates is fundamental in Edgeworth's "Method of Translation", and in certain theories which he has advanced concerning generalized frequency curves*.

It is a purpose of this paper to test the practicability of regarding some observed distributions as transformations of more elementary variates which follow the normal curve and to investigate the characters of some functions that are obtained by simple transformations of a normal distribution.

More precisely, if x_1, x_2, \dots, x_n are a set of elementary variates that are normally distributed, new variates X_1, X_2, \dots, X_n may be formed which are functions of these more elementary variates, say of the form

$$X_r = a_1 x_r + a_2 x_r^2 + \dots + a_n x_r^n + \dots \quad (1).$$

The X 's would of necessity follow some law of frequency. It may be of interest to determine the law for certain cases. The cases in which only a few terms of series (1) need be used are perhaps of the most interest.

If $a_2, a_3, \dots, a_n, \dots$ are small as compared with a_1 and we let $\kappa = \frac{a_2}{a_1}$; $\lambda = \frac{a_3}{a_1}$; \dots , we may write $X_r = a_1(x_r + \kappa x_r^2 + \lambda x_r^3 + \dots)$.

It is not usually necessary to go beyond the third power in x provided κ and λ are small decreasing numbers as they are in the numerical illustration of section V. Or conversely, having an observed distribution given, we shall inquire into the consequences of treating it as a distribution which is formed by transformation from a certain fundamental normal distribution.----

* Fifth International Congress of Mathematicians, vol. II, page 426, 1912. Journal of the Royal Statistical Society, 1898-1900.

1870

1. The first of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

2. The second of the year was a very wet one, and the crops were much injured by the rain. The weather was very cold, and the crops were much injured by the rain.

3. The third of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

4. The fourth of the year was a very wet one, and the crops were much injured by the rain. The weather was very cold, and the crops were much injured by the rain.

5. The fifth of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

6. The sixth of the year was a very wet one, and the crops were much injured by the rain. The weather was very cold, and the crops were much injured by the rain.

7. The seventh of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

8. The eighth of the year was a very wet one, and the crops were much injured by the rain. The weather was very cold, and the crops were much injured by the rain.

9. The ninth of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

10. The tenth of the year was a very wet one, and the crops were much injured by the rain. The weather was very cold, and the crops were much injured by the rain.

II. TRANSFORMATION $X = x^2$.

This is a very special type of transformation which surfaces might be expected to follow, since similar surfaces are to each other as the squares of their linear dimensions.

Let $f(X)dX$ be this distribution formed by squaring each variable of a system that follows the normal curve of standard deviation σ and center of gravity at $x=a$. Then the integral

$$\int f(X)dX = \frac{1}{\sigma\sqrt{2\pi}} \int e^{-\frac{(x-a)^2}{2\sigma^2}} dx$$

gives the probability that a variable x of the generating system falls between assigned limits of integration. Since $X=x^2$, $dX=2x dx$, and $dx = \frac{dX}{2\sqrt{X}}$. Substituting \sqrt{X} for x and $\frac{dX}{2\sqrt{X}}$ for dx in the above equation,

$$\int f(X)dX = \frac{1}{\sigma\sqrt{2\pi}} \int e^{-\frac{(\sqrt{X}-a)^2}{2\sigma^2}} \cdot \frac{1}{2\sqrt{X}} dX.$$

Since each element of the original group is squared to form the transformed distribution, each negative element of the normal distribution will become positive when squared. For this reason, the area under the normal curve is taken between the limits, say from $-\alpha$ to $+\alpha$, where α is any positive number, and the area under the transformed curve is taken from 0 to α^2 . By hypothesis, α is any positive number, and we may extend the limits to infinity for convenience of integration. Then

$$\frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty e^{-\frac{(\sqrt{X}-a)^2}{2\sigma^2}} \cdot \frac{1}{2\sqrt{X}} dX = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{(x-a)^2}{2\sigma^2}} dx.$$

As the unknown origin O is at a distance a from the center of gravity of the normal curve, a^2 is the distance from the median of the transformed curve to the same origin O , provided a is large enough so that none of the variates of the normal curve are below the origin.

Let g be the distance from the origin to the mean of the observed curve and let $\sigma\sqrt{2}$ be the modulus of the normal curve.

By method of moments we shall determine the three constants. Since each variate x of the normal curve is replaced by x' , we have

$$\begin{aligned}\int_0^{\infty} x^n f(x) dx &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^{\infty} x^n e^{-\frac{(\sqrt{x}-a)^2}{2\sigma^2}} \cdot \frac{1}{2\sqrt{x}} dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x'^n e^{-\frac{(x'-a)^2}{2\sigma^2}} dx,\end{aligned}$$

from which we see that we can use our knowledge of the normal curve in valuing the moments.

$$\begin{aligned}\text{First moment. } g &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x'^2 e^{-\frac{(x'-a)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x'+a)^2 e^{-\frac{x'^2}{2\sigma^2}} dx,\end{aligned}$$

when we put $x = x' + a$. Integrating by parts

$$g = a^2 + \sigma^2 \dots\dots (1).$$

The mean, therefore, is equal to the median plus σ^2 .

Second moment. The second moment coefficient about the center of gravity being denoted by μ_2 , that about the origin is,

$$g^2 + \mu_2 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x+a)^4 e^{-\frac{x^2}{2\sigma^2}} dx.$$

Integrating by parts gives $g^2 + \mu_2 = a^4 + 6a^2\sigma^2 + 3\sigma^4 \dots\dots (2)$

Third moment. The third moment coefficient about the center of gravity is μ_3 , that about the origin is,

$$\begin{aligned}g^3 + 3g\mu_2 + \mu_3 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x+a)^6 e^{-\frac{x^2}{2\sigma^2}} dx, \\ &= a^6 + 15a^4\sigma^2 + 45a^2\sigma^4 + 15\sigma^6 \dots\dots (3)\end{aligned}$$

By substituting the value of g from equation (1) into (2) and (3), we get

$$\mu_2 = 4a^2\sigma^2 + 2\sigma^4 \dots\dots (4)$$

$$\mu_3 = 24a^2\sigma^4 + 8\sigma^6 \dots\dots (5)$$

Eliminating a from (4) and (5)

$$4\sigma^6 - 6\sigma^4\mu_2 + \mu_3 = 0.$$



Substituting f for σ^2

$$4f^3 - 6f \cdot \mu_2 + \mu_3 = 0 \quad (6)$$

This cubic is easily solvable by Horner's method and the other constants may be determined from equations (4) and (5).

Geometric meaning of this transformation. One peculiarity of this type transformation is the infinite discontinuity at the origin. No matter what the modulus is or how far the center of gravity of the curve is removed from the origin, there is always, at least, one point of infinite discontinuity at the origin.

If we allow $\sigma=1$ and differentiate $\frac{1}{\sqrt{x}} e^{-\frac{(\sqrt{x}-a)^2}{2}}$ we shall be able to find the maxima, minima and points of inflection of the theoretical curve,

$$-\frac{1}{\sqrt{x}} e^{-\frac{(\sqrt{x}-a)^2}{2}} \cdot (\sqrt{x}-a) \cdot \frac{1}{2\sqrt{x}} + e^{-\frac{(\sqrt{x}-a)^2}{2}} \cdot \frac{1}{-2(\sqrt{x})^3} = 0, (\sqrt{x}-a) - \frac{1}{\sqrt{x}} = 0,$$

$$x = \frac{(a^2-2) \pm \sqrt{(a^2-2)^2-4}}{2}$$

We note that for $a < 2$ there is no maximum, minimum or point of inflection. For $a=2$ there is a point of inflection at $x=1$, but there is neither maximum nor minimum. For $a > 2$ we have one maximum and one minimum. For $a=4$, $x=13.92$ or 0.07 .

This indicates that it is necessary to have the origin back a distance at least 4σ in the negative direction from the origin in order that all the end values of the normal curve may be included in the positive field. If the origin is not back far enough, the negative end, on being squared, will distort the distribution in a most surprising way, as will be seen presently.

Figure 1 is the normal curve. In case $a=\sigma$ there is not the slightest resemblance between the elementary normal curve and the transformed curve

(fig. 6). When $x=0$, $y = \infty$. From this point there is a continuous curve without maximum, minimum or point of inflection.

When $a=2\sigma$ the curve is hardly recognizable as related to the normal curve, as shown in figure 5, there is neither maximum nor minimum. At $x=1$, there is a point of inflection with the inflectional tangent parallel to the x -axis.

Figure 7 illustrates the distortion for $a=3\sigma$. In this case the maximum is at 6.89 and the minimum at 0.11. Even here the origin is not back far enough.

If $a \geq 4\sigma$ the curve presents the general appearance of a much flattened normal curve as shown in figure 8.

It is therefore necessary, if we are to expect the transformed curve to be similar in general appearance to the fundamental normal curve, that the centroid of the normal curve must be at a distance of at least 4σ above the origin, otherwise our transformed curve will be utterly different in general appearance from the normal curve.

It might well be expected that the weights of 5082 men who are five feet, nine inches tall and are between the ages of twenty and twenty-four years of age would be distributed according to some transformed curve. I have accordingly selected an illustration from the Medico-Actuarial Mortality Investigation, vol. I, page 41.

Sheppard's corrections were applied to the moments calculated about the centroid because of high contact at the ends of the range.

If $\mu_1, \mu_2, \mu_3, \mu_4$ be written for the adjusted moments, it was found that

$$\mu_1 = 0$$

$$\mu_2 = 190.08565$$

$$\mu_3 = 2203.6126$$

$$\mu_4 = 174,557.2162$$

The mean is at 150.096418. The calculated theoretical normal frequency distribution is given in Table I.

The class mark of each frequency group is given in column I. The difference between the centroid and the class mark X is given in column II. Column II, divided by the standard deviation σ , is given in column III. The fractional area under the curve between 0 and $\frac{X-M}{\sigma}$, as it is tabulated in books on Probability*, is in column IV. The entire area is given in column V, the calculated frequency in column VI, the observed frequency in column VII, and the residuals in column VIII.

TABLE I.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.
97.5	52.5964	3.81487	0.49993	2541	1	1	0
102.5	47.5964	3.45222	0.49972	2539.5	3	...	3
107.5	42.5964	3.08956	0.49928	2537	12	1	11
112.5	37.5964	2.72691	0.49683	2525	30	2	28
117.5	32.5964	2.36425	0.49091	2495	69	26	43
122.5	27.5964	2.0016	0.47728	2426	142	94	48
127.5	22.5964	1.63894	0.44384	2284	256	248	8
132.5	17.5964	1.27628	0.39906	2028	404	488	84
137.5	12.5964	0.91363	0.31954	1624	561	724	163
142.5	7.5964	0.55097	0.20916	1063	683	746	63
147.5	2.5964	0.18832	0.07468	380	731	817	86
152.5	2.4036	0.17434	0.06919	351			
157.5	7.4036	0.53691	0.20432	1039	678	600	78
162.5	12.4036	0.89964	0.31584	1605	566	513	53
167.5	17.4036	1.26230	0.39657	2015	410	315	95

* Davenport, Statistical Methods, p. 119.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.
172.5	22.4036	1.62495	0.44805	2277	262	208	54
177.5	27.4036	1.987613	0.47657	2422	155	103	47
182.5	32.4036	2.35027	0.49062	2493	71	75	4
187.5	37.4036	2.71292	0.49663	2524	31	38	7
192.5	42.4036	3.07558	0.49895	2536	12	28	16
197.5	47.4036	3.43828	0.49971	2539.5	4	18	14
202.5	52.4036	3.80089	0.49993	2540.6	1	17	16
212.5	57.4036	4.1635	0.5	2541	.4	5	4.6
217.5	62.4036	Beyond the tables.				4	4.
222.5						2	2.
227.5						1	1.
232.5						2	2
237.5						..	0
242.5						..	0
247.5						1	1.
						..	0

The normal function does not describe this frequency distribution at all , as is obvious from the above table.

Before calculating the theoretical distribution in case of squares it is necessary to determine the values of the constants α and σ . Substituting the values of μ_2 and μ_3 into equation (3)

$$4f^3 - 6 \times 190.08565f + 2203.6126 = 0.$$

It is easily shown that this equation has three real roots of which two are positive and one is negative. The negative value would make α^2 negative, the larger positive value would make α smaller than σ . A distribution of this sort is shown in fig. VII. The central value is, therefore, the appropriate one.

$$f = 1.9585.$$

Name		Age		Sex		Profession		Marital Status		Religion		Education		Income		Assets		Liabilities		Total	
1	John	35	18	Male	Male	Teacher	Teacher	Married	Married	Protestant	Protestant	High School	High School	\$10,000	\$10,000	\$5,000	\$5,000	\$15,000	\$15,000	\$25,000	\$25,000
2	Mary	30	15	Female	Female	Homemaker	Homemaker	Married	Married	Catholic	Catholic	High School	High School	\$8,000	\$8,000	\$4,000	\$4,000	\$12,000	\$12,000	\$20,000	\$20,000
3	Robert	40	20	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$12,000	\$12,000	\$6,000	\$6,000	\$18,000	\$18,000	\$30,000	\$30,000
4	Elizabeth	38	18	Female	Female	Nurse	Nurse	Married	Married	Catholic	Catholic	College	College	\$9,000	\$9,000	\$5,000	\$5,000	\$14,000	\$14,000	\$24,000	\$24,000
5	William	45	25	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$15,000	\$15,000	\$8,000	\$8,000	\$23,000	\$23,000	\$40,000	\$40,000
6	Anna	32	16	Female	Female	Teacher	Teacher	Married	Married	Catholic	Catholic	High School	High School	\$7,000	\$7,000	\$3,000	\$3,000	\$10,000	\$10,000	\$17,000	\$17,000
7	Charles	42	22	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$11,000	\$11,000	\$6,000	\$6,000	\$17,000	\$17,000	\$29,000	\$29,000
8	Margaret	36	17	Female	Female	Homemaker	Homemaker	Married	Married	Catholic	Catholic	High School	High School	\$6,000	\$6,000	\$2,000	\$2,000	\$8,000	\$8,000	\$14,000	\$14,000
9	Thomas	48	28	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$16,000	\$16,000	\$9,000	\$9,000	\$25,000	\$25,000	\$44,000	\$44,000
10	Isabella	34	19	Female	Female	Nurse	Nurse	Married	Married	Catholic	Catholic	College	College	\$8,000	\$8,000	\$4,000	\$4,000	\$12,000	\$12,000	\$22,000	\$22,000
11	James	44	24	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$13,000	\$13,000	\$7,000	\$7,000	\$20,000	\$20,000	\$33,000	\$33,000
12	Frances	37	18	Female	Female	Homemaker	Homemaker	Married	Married	Catholic	Catholic	High School	High School	\$5,000	\$5,000	\$1,000	\$1,000	\$6,000	\$6,000	\$11,000	\$11,000
13	George	46	26	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$17,000	\$17,000	\$10,000	\$10,000	\$27,000	\$27,000	\$47,000	\$47,000
14	Julia	33	17	Female	Female	Teacher	Teacher	Married	Married	Catholic	Catholic	High School	High School	\$6,000	\$6,000	\$2,000	\$2,000	\$8,000	\$8,000	\$14,000	\$14,000
15	Edward	47	27	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$14,000	\$14,000	\$8,000	\$8,000	\$22,000	\$22,000	\$36,000	\$36,000
16	Charlotte	39	19	Female	Female	Nurse	Nurse	Married	Married	Catholic	Catholic	College	College	\$7,000	\$7,000	\$3,000	\$3,000	\$10,000	\$10,000	\$17,000	\$17,000
17	Henry	49	29	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$18,000	\$18,000	\$11,000	\$11,000	\$29,000	\$29,000	\$49,000	\$49,000
18	Emily	35	18	Female	Female	Homemaker	Homemaker	Married	Married	Catholic	Catholic	High School	High School	\$4,000	\$4,000	\$1,000	\$1,000	\$5,000	\$5,000	\$10,000	\$10,000
19	Frederick	50	30	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$19,000	\$19,000	\$12,000	\$12,000	\$31,000	\$31,000	\$51,000	\$51,000
20	Harriet	41	21	Female	Female	Teacher	Teacher	Married	Married	Catholic	Catholic	High School	High School	\$5,000	\$5,000	\$1,000	\$1,000	\$6,000	\$6,000	\$11,000	\$11,000
21	Isaac	51	31	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$20,000	\$20,000	\$13,000	\$13,000	\$33,000	\$33,000	\$53,000	\$53,000
22	Josephine	43	23	Female	Female	Nurse	Nurse	Married	Married	Catholic	Catholic	College	College	\$6,000	\$6,000	\$2,000	\$2,000	\$8,000	\$8,000	\$14,000	\$14,000
23	Samuel	52	32	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$21,000	\$21,000	\$14,000	\$14,000	\$35,000	\$35,000	\$55,000	\$55,000
24	Abigail	45	25	Female	Female	Homemaker	Homemaker	Married	Married	Catholic	Catholic	High School	High School	\$3,000	\$3,000	\$1,000	\$1,000	\$4,000	\$4,000	\$9,000	\$9,000
25	Benjamin	53	33	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$22,000	\$22,000	\$15,000	\$15,000	\$37,000	\$37,000	\$57,000	\$57,000
26	Clara	47	27	Female	Female	Teacher	Teacher	Married	Married	Catholic	Catholic	High School	High School	\$4,000	\$4,000	\$1,000	\$1,000	\$5,000	\$5,000	\$10,000	\$10,000
27	David	54	34	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$23,000	\$23,000	\$16,000	\$16,000	\$39,000	\$39,000	\$59,000	\$59,000
28	Eleanor	49	29	Female	Female	Nurse	Nurse	Married	Married	Catholic	Catholic	College	College	\$5,000	\$5,000	\$2,000	\$2,000	\$7,000	\$7,000	\$12,000	\$12,000
29	Frank	55	35	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$24,000	\$24,000	\$17,000	\$17,000	\$41,000	\$41,000	\$61,000	\$61,000
30	Grace	51	31	Female	Female	Homemaker	Homemaker	Married	Married	Catholic	Catholic	High School	High School	\$2,000	\$2,000	\$1,000	\$1,000	\$3,000	\$3,000	\$7,000	\$7,000
31	Harold	56	36	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$25,000	\$25,000	\$18,000	\$18,000	\$43,000	\$43,000	\$63,000	\$63,000
32	Irene	53	33	Female	Female	Teacher	Teacher	Married	Married	Catholic	Catholic	High School	High School	\$1,000	\$1,000	\$1,000	\$1,000	\$2,000	\$2,000	\$6,000	\$6,000
33	John	57	37	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$26,000	\$26,000	\$19,000	\$19,000	\$45,000	\$45,000	\$65,000	\$65,000
34	Katherine	55	35	Female	Female	Nurse	Nurse	Married	Married	Catholic	Catholic	College	College	\$3,000	\$3,000	\$2,000	\$2,000	\$5,000	\$5,000	\$10,000	\$10,000
35	Leo	58	38	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$27,000	\$27,000	\$20,000	\$20,000	\$47,000	\$47,000	\$67,000	\$67,000
36	Margaret	59	39	Female	Female	Homemaker	Homemaker	Married	Married	Catholic	Catholic	High School	High School	\$1,000	\$1,000	\$1,000	\$1,000	\$2,000	\$2,000	\$6,000	\$6,000
37	Nathan	59	39	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$28,000	\$28,000	\$21,000	\$21,000	\$49,000	\$49,000	\$69,000	\$69,000
38	Olivia	61	41	Female	Female	Teacher	Teacher	Married	Married	Catholic	Catholic	High School	High School	\$1,000	\$1,000	\$1,000	\$1,000	\$2,000	\$2,000	\$6,000	\$6,000
39	Philip	60	40	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$29,000	\$29,000	\$22,000	\$22,000	\$51,000	\$51,000	\$71,000	\$71,000
40	Rebecca	63	43	Female	Female	Nurse	Nurse	Married	Married	Catholic	Catholic	College	College	\$2,000	\$2,000	\$2,000	\$2,000	\$4,000	\$4,000	\$8,000	\$8,000
41	Samuel	64	44	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$30,000	\$30,000	\$23,000	\$23,000	\$53,000	\$53,000	\$73,000	\$73,000
42	Theresa	65	45	Female	Female	Homemaker	Homemaker	Married	Married	Catholic	Catholic	High School	High School	\$1,000	\$1,000	\$1,000	\$1,000	\$2,000	\$2,000	\$6,000	\$6,000
43	Uriah	66	46	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$31,000	\$31,000	\$24,000	\$24,000	\$55,000	\$55,000	\$75,000	\$75,000
44	Virginia	67	47	Female	Female	Teacher	Teacher	Married	Married	Catholic	Catholic	High School	High School	\$1,000	\$1,000	\$1,000	\$1,000	\$2,000	\$2,000	\$6,000	\$6,000
45	William	68	48	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$32,000	\$32,000	\$25,000	\$25,000	\$57,000	\$57,000	\$77,000	\$77,000
46	Xenia	69	49	Female	Female	Nurse	Nurse	Married	Married	Catholic	Catholic	College	College	\$2,000	\$2,000	\$2,000	\$2,000	\$4,000	\$4,000	\$8,000	\$8,000
47	Yakov	70	50	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$33,000	\$33,000	\$26,000	\$26,000	\$59,000	\$59,000	\$79,000	\$79,000
48	Zoe	71	51	Female	Female	Homemaker	Homemaker	Married	Married	Catholic	Catholic	High School	High School	\$1,000	\$1,000	\$1,000	\$1,000	\$2,000	\$2,000	\$6,000	\$6,000
49	Adrian	72	52	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$34,000	\$34,000	\$27,000	\$27,000	\$61,000	\$61,000	\$81,000	\$81,000
50	Bella	73	53	Female	Female	Teacher	Teacher	Married	Married	Catholic	Catholic	High School	High School	\$1,000	\$1,000	\$1,000	\$1,000	\$2,000	\$2,000	\$6,000	\$6,000
51	Calvin	74	54	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$35,000	\$35,000	\$28,000	\$28,000	\$63,000	\$63,000	\$83,000	\$83,000
52	Dora	75	55	Female	Female	Nurse	Nurse	Married	Married	Catholic	Catholic	College	College	\$2,000	\$2,000	\$2,000	\$2,000	\$4,000	\$4,000	\$8,000	\$8,000
53	Ernest	76	56	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$36,000	\$36,000	\$29,000	\$29,000	\$65,000	\$65,000	\$85,000	\$85,000
54	Florence	77	57	Female	Female	Homemaker	Homemaker	Married	Married	Catholic	Catholic	High School	High School	\$1,000	\$1,000	\$1,000	\$1,000	\$2,000	\$2,000	\$6,000	\$6,000
55	Gustav	78	58	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$37,000	\$37,000	\$30,000	\$30,000	\$67,000	\$67,000	\$87,000	\$87,000
56	Helen	79	59	Female	Female	Teacher	Teacher	Married	Married	Catholic	Catholic	High School	High School	\$1,000	\$1,000	\$1,000	\$1,000	\$2,000	\$2,000	\$6,000	\$6,000
57	Isaac	80	60	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$38,000	\$38,000	\$31,000	\$31,000	\$69,000	\$69,000	\$89,000	\$89,000
58	Jessie	81	61	Female	Female	Nurse	Nurse	Married	Married	Catholic	Catholic	College	College	\$2,000	\$2,000	\$2,000	\$2,000	\$4,000	\$4,000	\$8,000	\$8,000
59	Kenneth	82	62	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$39,000	\$39,000	\$32,000	\$32,000	\$71,000	\$71,000	\$91,000	\$91,000
60	Lillian	83	63	Female	Female	Homemaker	Homemaker	Married	Married	Catholic	Catholic	High School	High School	\$1,000	\$1,000	\$1,000	\$1,000	\$2,000	\$2,000	\$6,000	\$6,000
61	Martin	84	64	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$40,000	\$40,000	\$33,000	\$33,000	\$73,000	\$73,000	\$93,000	\$93,000
62	Nancy	85	65	Female	Female	Teacher	Teacher	Married	Married	Catholic	Catholic	High School	High School	\$1,000	\$1,000	\$1,000	\$1,000	\$2,000	\$2,000	\$6,000	\$6,000
63	Oscar	86	66	Male	Male	Engineer	Engineer	Married	Married	Protestant	Protestant	College	College	\$41,000	\$41,000	\$34,000	\$34,000	\$75,000	\$75,000	\$95,000	\$95,000
64	Pamela	87	67	Female	Female	Nurse	Nurse	Married	Married	Catholic	Catholic	College	College	\$2,000	\$2,000	\$2,000	\$2,000	\$4,000	\$4,000	\$8,000	\$8,000
65	Quentin	88	68	Male	Male	Businessman	Businessman	Married	Married	Protestant	Protestant	College	College	\$42,000	\$42,000	\$35,000	\$35,000	\$77,000	\$77,000	\$97,000	\$97,000
66	Ruth	89	69	Female	Female	Homemaker	Homemaker	Married	Married	Catholic	Catholic	High School	High School	\$1,000	\$1,000	\$1,000	\$1,000</				

Since $f = \sigma^2$

$$\sigma = \pm 1.3987.$$

Substituting the value of σ in equation (4)

$$a^2 = 23.74563,$$

$$a = \pm 4.8759.$$

With a no larger as compared to σ we could hardly expect an excellent fit.

In table II, an attempt was made to fit this observed distribution by means of a function where $X=x^2$, x 's being the variates of the theoretical normal distribution. Column I contains the class marks; col. II, the distance from the origin to the median of the transformed curve plus the deviation from that median, a^2+X-M ; col. III, $\frac{\sqrt{a^2+X-M}-a}{\sigma} = \frac{x}{\sigma}$; col. IV, the fractional area between 0 and $\frac{x}{\sigma}$ *; col. V, fractional area times the number of frequencies; col. VI, the calculated frequencies; col. VII, the observed frequencies, and col. VIII the residuals.

TABLE II.

Above the median.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.
152.5	23.1367	0.30635	0.12032	611.4	717.	817.	100.
157.5	33.1367	0.62591	0.23431	1191.	580.	600.	20.
162.5	38.1367	0.92912	0.32357	1644.	453.	513.	60.
167.5	43.1367	1.20967	0.38365	1948.	304.	315.	11.
172.5	48.1367	1.4743	0.42979	2184.	236.	208.	28.
177.5	53.1367	1.7256	0.45778	2326.	142.	108.	34.
182.5	58.1367	1.96529	0.47530	2415.	89.	75.	14.
187.5	63.1367	2.19482	0.48591	2469.	54.	38.	16.
192.5	68.1367	2.4157	0.49214	2501.	32.	28.	4.
197.5	73.1367	2.62815	0.49569	2519.	18.	18.	0.
202.5	78.1367	2.8337	0.49769	2529.	10.	17.	7.

* Davenport, Statistical methods, pages 119-125.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.
207.5	83.1367	3.03274	0.49880	2535.	6.	5.	1.
212.5	88.1367	3.22592	0.49937	2537.	2.	4.	2.
217.5	93.1367	3.41355	0.49968	2539.	2.	2.	0.
222.5	98.1367	3.59648	0.49983	2540.	1.	1.	0.
227.5	103.1367	3.7749	0.49992	2540.5	0.5	2.	1.5
232.5	108.1367	3.94702	0.49996	2540.7	0.2	..	0.2
237.5	113.1367	4.11732	Beyond the tables.			..	0.
						1.	1.

Below the median.

147.5	23.1367	0.04704	0.01875	96.			
142.5	18.1367	0.44126	0.17048	866.	770.	746.	24.
137.5	13.1367	0.96625	0.33302	1692.	826.	724.	102.
132.5	8.1367	1.4467	0.425593	2163.	471.	438.	17.
127.5	3.1367	2.21984	0.48679	2474.	311.	248.	63.
122.5	-1.9633	imaginary.				94.	94.
						26.	26.
						2.	2.
						1.	1.
						0.	0.
						1.	1.

This transformation describes the upper end of the distribution fairly well, toward the center the transformed curve is too low and below the median there is no fit, the values becoming imaginary after the fourth. The appearance of the imaginary variates at the lower end of the distribution shows that the origin is not back far enough, in other words, that part of the variates of the elementary normal distribution fall in the negative region. After squaring these variates they become positive and change the form of the curve.

1870		1871		1872		1873		1874		1875		1876		1877		1878		1879		1880		1881		1882		1883		1884		1885		1886		1887		1888		1889		1890		1891		1892		1893		1894		1895		1896		1897		1898		1899		1900		1901		1902		1903		1904		1905		1906		1907		1908		1909		1910		1911		1912		1913		1914		1915		1916		1917		1918		1919		1920		1921		1922		1923		1924		1925		1926		1927		1928		1929		1930		1931		1932		1933		1934		1935		1936		1937		1938		1939		1940		1941		1942		1943		1944		1945		1946		1947		1948		1949		1950		1951		1952		1953		1954		1955		1956		1957		1958		1959		1960		1961		1962		1963		1964		1965		1966		1967		1968		1969		1970		1971		1972		1973		1974		1975		1976		1977		1978		1979		1980		1981		1982		1983		1984		1985		1986		1987		1988		1989		1990		1991		1992		1993		1994		1995		1996		1997		1998		1999		2000		2001		2002		2003		2004		2005		2006		2007		2008		2009		2010		2011		2012		2013		2014		2015		2016		2017		2018		2019		2020		2021		2022		2023		2024		2025		2026		2027		2028		2029		2030		2031		2032		2033		2034		2035		2036		2037		2038		2039		2040		2041		2042		2043		2044		2045		2046		2047		2048		2049		2050		2051		2052		2053		2054		2055		2056		2057		2058		2059		2060		2061		2062		2063		2064		2065		2066		2067		2068		2069		2070		2071		2072		2073		2074		2075		2076		2077		2078		2079		2080		2081		2082		2083		2084		2085		2086		2087		2088		2089		2090		2091		2092		2093		2094		2095		2096		2097		2098		2099		2100		2101		2102		2103		2104		2105		2106		2107		2108		2109		2110		2111		2112		2113		2114		2115		2116		2117		2118		2119		2120		2121		2122		2123		2124		2125		2126		2127		2128		2129		2130		2131		2132		2133		2134		2135		2136		2137		2138		2139		2140		2141		2142		2143		2144		2145		2146		2147		2148		2149		2150		2151		2152		2153		2154		2155		2156		2157		2158		2159		2160		2161		2162		2163		2164		2165		2166		2167		2168		2169		2170		2171		2172		2173		2174		2175		2176		2177		2178		2179		2180		2181		2182		2183		2184		2185		2186		2187		2188		2189		2190		2191		2192		2193		2194		2195		2196		2197		2198		2199		2200		2201		2202		2203		2204		2205		2206		2207		2208		2209		2210		2211		2212		2213		2214		2215		2216		2217		2218		2219		2220		2221		2222		2223		2224		2225		2226		2227		2228		2229		2230		2231		2232		2233		2234		2235		2236		2237		2238		2239		2240		2241		2242		2243		2244		2245		2246		2247		2248		2249		2250		2251		2252		2253		2254		2255		2256		2257		2258		2259		2260		2261		2262		2263		2264		2265		2266		2267		2268		2269		2270		2271		2272		2273		2274		2275		2276		2277		2278		2279		2280		2281		2282		2283		2284		2285		2286		2287		2288		2289		2290		2291		2292		2293		2294		2295		2296		2297		2298		2299		2300		2301		2302		2303		2304		2305		2306		2307		2308		2309		2310		2311		2312		2313		2314		2315		2316		2317		2318		2319		2320		2321		2322		2323		2324		2325		2326		2327		2328		2329		2330		2331		2332		2333		2334		2335		2336		2337		2338		2339		2340		2341		2342		2343		2344		2345		2346		2347		2348		2349		2350		2351		2352		2353		2354		2355		2356		2357		2358		2359		2360		2361		2362		2363		2364		2365		2366		2367		2368		2369		2370		2371		2372		2373		2374		2375		2376		2377		2378		2379		2380		2381		2382		2383		2384		2385		2386		2387		2388		2389		2390		2391		2392		2393		2394		2395		2396		2397		2398		2399		2400		2401		2402		2403		2404		2405		2406		2407		2408		2409		2410		2411		2412		2413		2414		2415		2416		2417		2418		2419		2420		2421		2422		2423		2424		2425		2426		2427		2428		2429		2430		2431		2432		2433		2434		2435		2436		2437		2438		2439		2440		2441		2442		2443		2444		2445		2446		2447		2448		2449		2450		2451		2452		2453		2454		2455		2456		2457		2458		2459		2460		2461		2462		2463		2464		2465		2466		2467		2468		2469		2470		2471		2472		2473		2474		2475		2476		2477		2478		2479		2480		2481		2482		2483		2484		2485		2486		2487		2488		2489		2490		2491		2492		2493		2494		2495		2496		2497		2498		2499		2500		2501		2502		2503		2504		2505		2506		2507		2508		2509		2510		2511		2512		2513		2514		2515		2516		2517		2518		2519		2520		2521		2522		2523		2524		2525		2526		2527		2528		2529		2530		2531		2532		2533		2534		2535		2536		2537		2538		2539		2540		2541		2542		2543		2544		2545		2546		2547		2548		2549		2550		2551		2552		2553		2554		2555		2556		2557		2558		2559		2560		2561		2562		2563		2564		2565		2566		2567		2568		2569		2570		2571		2572		2573		2574		2575		2576		2577		2578		2579		2580		2581		2582		2583		2584		2585		2586		2587		2588		2589		2590		2591		2592		2593		2594		2595		2596		2597		2598		2599		2600		2601		2602		2603		2604		2605		2606		2607		2608		2609		2610		2611		2612		2613		2614		2615		2616		2617		2618		2619		2620		2621		2622		2623		2624		2625		2626		2627		2628		2629		2630		2631		2632		2633		2634		2635		2636		2637		2638		2639		2640		2641		2642		2643		2644		2645		2646		2647		2648		2649		2650		2651		2652		2653		2654		2655		2656		2657		2658		2659		2660		2661		2662		2663		2664		2665		2666		2667		2668		2669		2670		2671		2672		2673		2674		2675		2676		2677		2678		2679		2680		2681		2682		2683		2684		2685		2686		2687		2688		2689		2690		2691		2692		2693		2694		2695		2696		2697		2698		2699		2700		2701		2702		27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III. TRANSFORMATION $X = x^3$.

In case of similar volumes it seems quite reasonable that we might have a distribution $X = x^3$, where each variate of the normal distribution is cubed. In case the variates distribute themselves in this fashion, the constants may be determined by method of moments, in a manner similar to that shown in section II.

$$\int_{-\infty}^{+\infty} X^n f(X) dX = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^{3n} e^{-\frac{(x-a)^2}{2\sigma^2}} dx.$$

The limits of the observed function may be taken from $-\infty$ to $+\infty$ as the cubes of the negative numbers remain negative.

First moment.
$$\bar{X} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^3 e^{-\frac{(x-a)^2}{2\sigma^2}} dx$$

Let $x=x'+a$
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x+a)^3 e^{-\frac{x'^2}{2\sigma^2}} dx$$

$$= 3a\sigma^2 + a^3$$

Second moment.

$$\begin{aligned} \bar{X}^2 + \mu_2 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x+a)^6 e^{-\frac{x'^2}{2\sigma^2}} dx \\ &= a^6 + 15a^4\sigma^2 + 45a^2\sigma^4 + 15\sigma^6. \end{aligned}$$

Third moment.

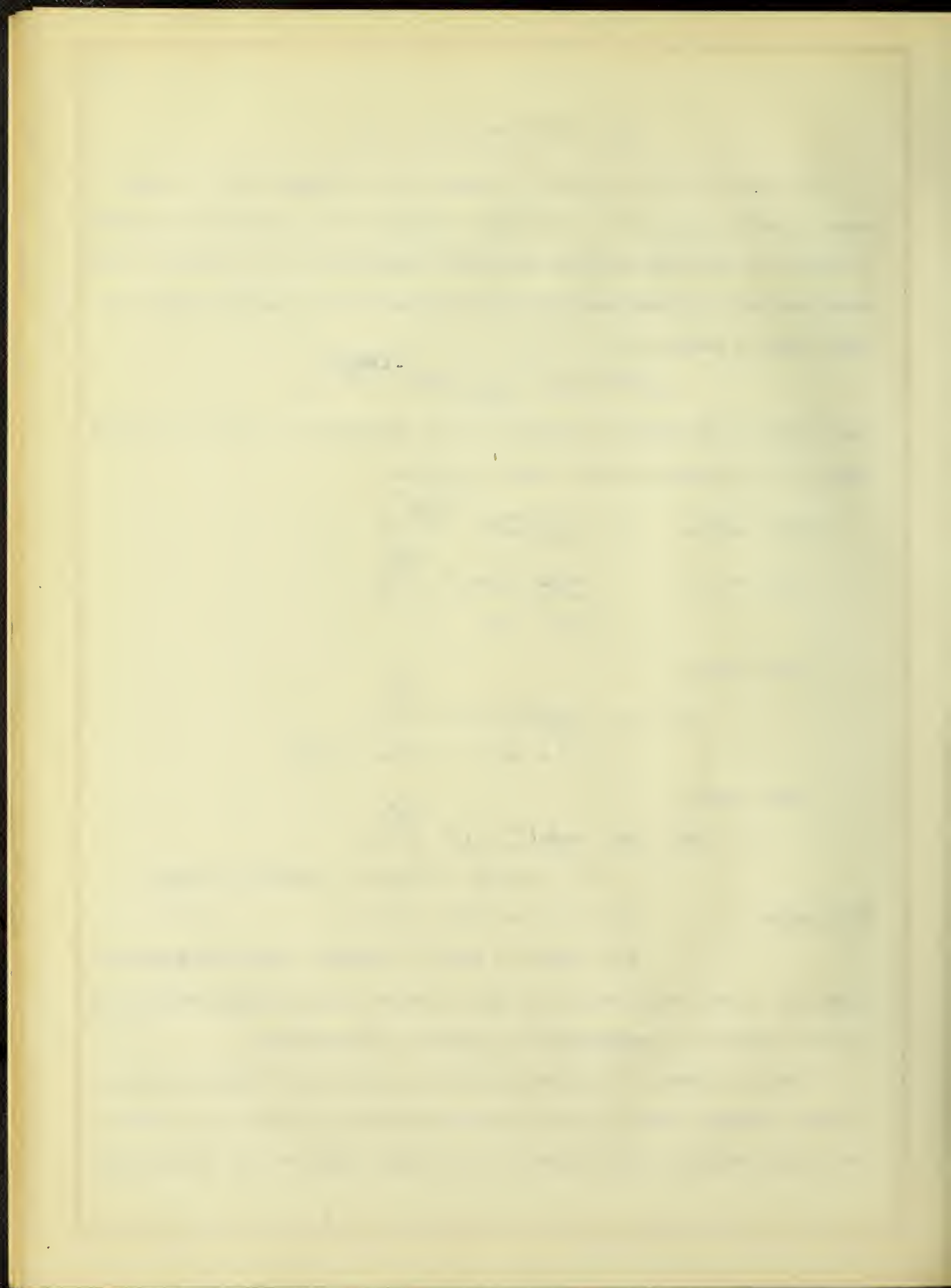
$$\begin{aligned} \bar{X}^3 + 3\bar{X}\mu_2 + \mu_3 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x+a)^9 e^{-\frac{x'^2}{2\sigma^2}} dx \\ &= a^9 + 36a^7\sigma^2 + 378a^5\sigma^4 + 1260a^3\sigma^6 + 945a\sigma^8. \end{aligned}$$

Eliminating \bar{X}
$$\mu_2 = 3a^4\sigma^2 + 9a^2\sigma^4 + 15\sigma^6.$$

$$\mu_3 = 945a\sigma^8 + 99a^3\sigma^6 + 351a^5\sigma^4 + 27a^7\sigma^2 - 3a\mu_2(3\sigma^2+a^2).$$

Equations of this sort are of too high a degree and too complicated to be of use in making an application to observed distributions.

Since the cubes of negative numbers are negative it would, perhaps, at first thought, seem that the position of the origin would not effect the transformation. This, however, is not true. When $x=0$ the transformed



curve

$$y = \frac{1}{3\sigma\sqrt{X^2/2\pi}} \cdot e^{-\frac{(\sqrt{X})^2}{2\sigma^2}}$$

is similar in some respects to

$$y = \frac{1}{2\sigma\sqrt{2\pi X}} \cdot e^{-\frac{(\sqrt{X})^2}{2\sigma^2}}$$

Both of these curves begin at ∞ when $x=0$, then descend very rapidly for a time, touching the x -axis at infinity. The fundamental difference between the two cases is that the case of cubes has a pair of curves beginning at infinity for $x=0$ and extending in opposite directions and intersecting the x -axis at $-\infty$ and $+\infty$. Figure no. 2 is a graph of this curve. It will be noted that in the case of squares there is a single curve similar in shape to this curve in the first quadrant.

It is necessary that $a > \sigma\sqrt{3}$ in order that we may have a maximum or a minimum on either side of the y -axis. This function, as well as the function discussed in section II, has an infinite discontinuity at the origin. With regard to this infinite discontinuity, it does not seem likely that there would be any analogy in nature.

Let us determine the maxima and minima of

$$y = \frac{1}{3\sqrt{X^2/2\pi}} \cdot e^{-\frac{(\sqrt{X}-a)^2}{2}}$$

$$\frac{dy}{dX} = X^{-\frac{3}{2}} \cdot e^{-\frac{(\sqrt{X}-a)^2}{2}} \left[-(\sqrt{X}-a)X^{-\frac{1}{2}} + e^{-\frac{(\sqrt{X}-a)^2}{2}} \left(-\frac{2\sqrt{X}-5}{3} \right) \right] = 0,$$

$$X^{\frac{2}{3}} - aX^{\frac{1}{3}} + 2 = 0,$$

$$X^{\frac{1}{3}} = \frac{a \mp \sqrt{a^2 - 8}}{2}.$$

For $a=\sigma$ or $a=2\sigma$, X is imaginary, there is, therefore, no maxima, minima or point of inflection. See figs. no. 3 and 4. For $a=3\sigma$, $X=1$ or 8 making a minimum at 1 and a maximum at 8. If the variates of the elementary normal curve are all positive, there will be a smooth flat curve, similar in form to the normal curve.

IV. TRANSFORMATION $X = a_1x + a_2x^2$

This is a more general transformation than those treated in sections II and III. There is a certain propriety in using the first two or three terms of a Taylor's expansion. In this section, we will consider only the first two terms of the expansion. It is necessary that a_2 shall be small as compared to a_1 , otherwise the curve will be very much distorted from a normal curve, since the theoretical distribution is here calculated from the median of the observed distribution.

It is equally general to make $\sigma = \frac{1}{\sqrt{2}}$ or $\sigma\sqrt{2}=1$, and to replace $\frac{x}{\sigma\sqrt{2}}$ by E , thus using Edgeworth's notation. Since by hypothesis a_2 is small compared to a_1 , let $\frac{a_2}{a_1} = \kappa$ where κ is small. Then

$$X = a(E + \kappa E^2)$$

and

$$dX = a(1 + 2\kappa E)dE;$$

then

$$\int_{-\infty}^{+\infty} e^{-E^2} dE = \int e^{-\frac{1}{4\kappa} \frac{1+2\kappa E}{a(1+2\kappa(-1\pm\sqrt{1+\frac{4\kappa X}{a}}))}} dX.$$

We shall denote the moments about the median as M_1, M_2, M_3, M_4 respectively, and $\mu_1, \mu_2, \mu_3, \mu_4$ denote the moments about the center of gravity.

$$\begin{aligned} \text{First moment. } M_1 &= \int_{-\infty}^{+\infty} a(E + \kappa E^2)e^{-E^2} dE \\ &= \frac{1}{2}a\kappa. \end{aligned}$$

$$\begin{aligned} \text{Second moment. } M_2 &= \int_{-\infty}^{+\infty} a^2(E + \kappa E^2)^2 e^{-E^2} dE \\ &= \frac{1}{2}a^2 + \frac{3}{4}a^2\kappa^2. \end{aligned}$$

$$\begin{aligned} \text{Third moment. } M_3 &= \int_{-\infty}^{+\infty} a^3(E + \kappa E^2)^3 e^{-E^2} dE \\ &= \frac{9}{4}\kappa a^3 + \frac{15}{8}\kappa^3 a^3. \end{aligned}$$

$$\begin{aligned} \mu_2 &= M_2 - M_1^2 = \frac{1}{2}a^2 + \frac{3}{4}a^2\kappa^2 - \frac{1}{4}a^2\kappa^2 \\ &= \frac{1}{2}a^2(1 + \kappa^2). \end{aligned}$$

$$\mu_3 = M_3 - 3M_1\mu_2 - M_1^3 = a^3\kappa\left(\frac{3}{2} + \kappa^2\right).$$

$$\beta_1^* = \frac{\mu_2^2}{\mu_3^2} = \frac{[a^3 \kappa (\frac{9}{2} + \kappa^2)]^2}{[\frac{1}{2} a^2 (1 + \kappa^2)]^3} \quad (1)$$

$$= \frac{a^6 \kappa^2 (\frac{9}{2} + \kappa^2 + 3\kappa^2 + \kappa^4)}{a^6 \frac{1}{8} (1 + 3\kappa^2 + 3\kappa^4 + \kappa^6)}.$$

From (1) $\kappa^2 (\frac{9}{2} + 3\kappa^2 + \kappa^4) - \beta_1 \frac{1}{8} (1 + 3\kappa^2 + 3\kappa^4 + \kappa^6) = 0.$

Let $\chi = \kappa^2$. Then

$$8\chi (\frac{9}{2} + 3\chi + \chi^2) - \beta_1 (1 + 3\chi + 3\chi^2 + \chi^3) = 0,$$

$$18\chi + 24\chi^2 + 8\chi^3 - \beta_1 (1 + 3\chi + 3\chi^2 + \chi^3) = 0. \quad (2)$$

$$X = a(\xi + \kappa \xi^2),$$

$$\xi = \frac{-1 \pm \sqrt{1 + \frac{4\kappa X}{a}}}{2\kappa} \quad (3).$$

Since $\xi = \frac{X}{a\sqrt{2}}$, it is highly improbable for $|\xi|$ to be greater than 5. In fact we shall prove presently, that negative values of ξ lie between 0 and -5. This transformation will be applied to the data used in section II. To do this it is necessary to get first the values of a and κ .

From (1) $\beta = \frac{\mu_2^2}{\mu_3^2} = \frac{4855908.4908}{6868280.078} = 0.707005.$

$$7.292995\chi^3 + 21.878985\chi^2 + 15.878985\chi - 0.707005 = 0,$$

$$\chi^3 + 3\chi^2 + 2.177292\chi - 0.096943 = 0,$$

$$\chi = 0.0418.$$

$$\kappa^2 = \chi,$$

$$\kappa = \pm 0.2044504.$$

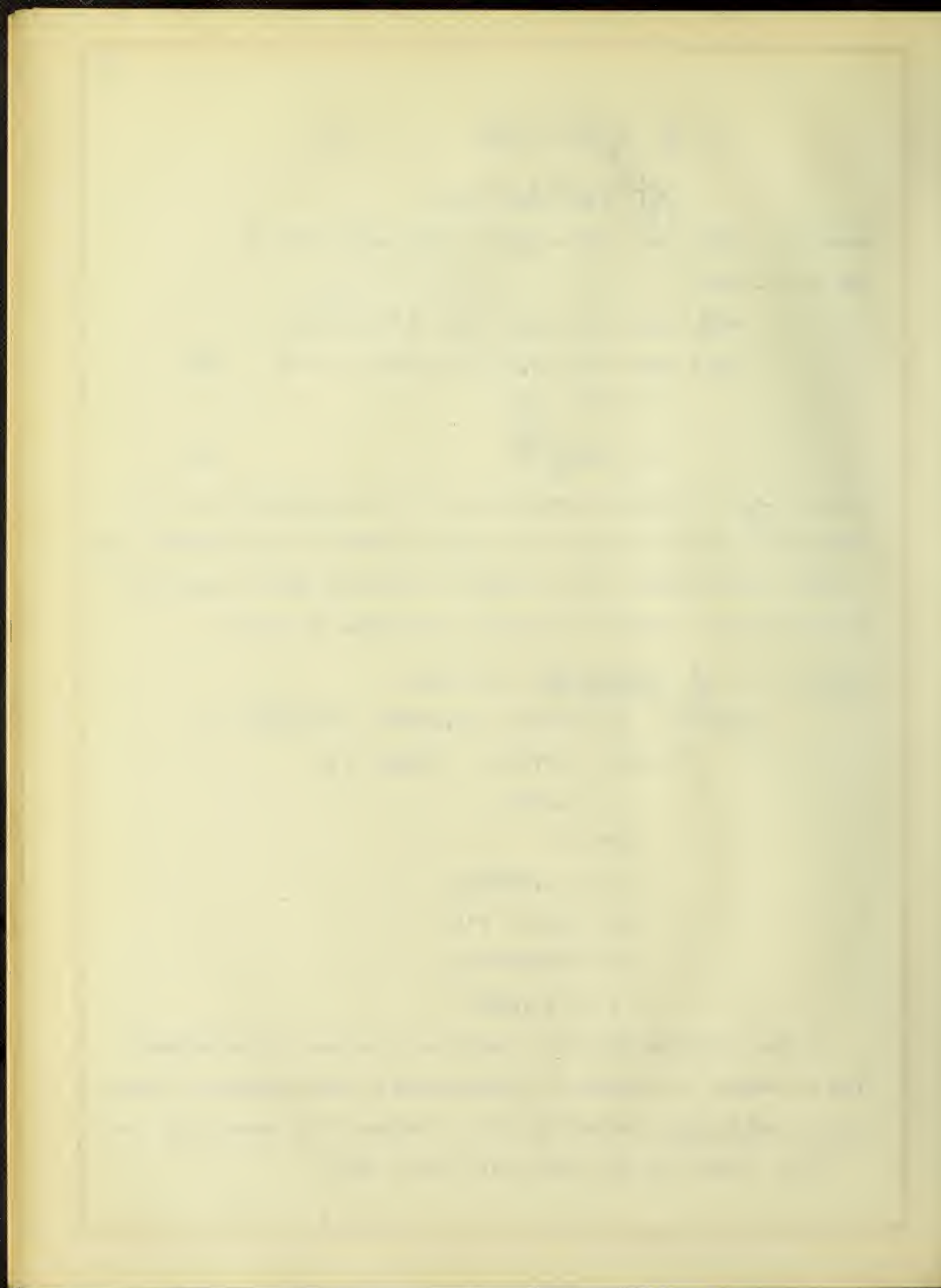
$$\mu_2 = \frac{1}{2} a^2 (1 + \kappa^2),$$

$$a^2 = 364.9177385,$$

$$a = \pm 19.1028.$$

In the following table III, X as given in column I is calculated from the median, ξ of column II is calculated by substituting the values of X , a , and κ into equation (3), $\phi(\xi)$ of column III is taken from Czu-

* Phil. Trans. Roy. Soc. 1895, vol. 186, p. 351.



ber*. $\varphi(\xi) \times \frac{5082}{2}$ is found in column IV. The calculated theoretical frequency is given in column V, the observed frequency, in column VI, and the residuals, in column VII.

TABLE III.

Above the median.

I.	II.	III.	IV.	V.	VI.	VII.
3.7087	0.187014	0.208601	530	729	817	90
8.7087	0.419821	0.44732	1136	606	600	6
13.7087	0.635216	0.63128	1604	468	513	45
18.7087	0.836317	0.76308	1939	335	315	20
23.7087	1.022475	0.851738	2164	225	208	17
28.7087	1.20572	0.91181	2317	153	108	45
33.7087	1.376364	0.94846	2410	93	75	18
38.7087	1.540841	0.97068	2466	56	38	18
43.7087	1.698141	0.98368	2499	33	28	5
48.7087	1.849278	0.99106	2518	19	18	1
53.7087	1.996378	0.99522	2529	11	17	6
58.7087	2.138178	0.99748	2534	5	5	0
63.7087	2.275739	0.998705	2538	4	4	0
68.7087	2.409378	0.999343	2539	1+	2	1
73.7087	2.539985	0.999670	2540	1	1	0
78.7087	2.66654	0.999837	2541.-	1-	2	1
83.7087	2.78974	0.9999199	2541.-	0	0	0
88.7087				0	0	0
93.7087				0	1	1

Below the median.

1.2913	0.06857	0.077627	197			
6.2913	0.355268	0.384668	977	780	746	34
11.2913	0.687809	0.66926	1700	723	724	1

* Wahrscheinlichkeitsrechnung, page 385.

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 DEPARTMENT OF CHEMISTRY

NAME		RESIDENCE		DATE	
1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96
97	98	99	100	101	102
103	104	105	106	107	108
109	110	111	112	113	114
115	116	117	118	119	120
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127	128	129	130	131	132
133	134	135	136	137	138
139	140	141	142	143	144
145	146	147	148	149	150
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157	158	159	160	161	162
163	164	165	166	167	168
169	170	171	172	173	174
175	176	177	178	179	180
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193	194	195	196	197	198
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205	206	207	208	209	210
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217	218	219	220	221	222
223	224	225	226	227	228
229	230	231	232	233	234
235	236	237	238	239	240
241	242	243	244	245	246
247	248	249	250	251	252
253	254	255	256	257	258
259	260	261	262	263	264
265	266	267	268	269	270
271	272	273	274	275	276
277	278	279	280	281	282
283	284	285	286	287	288
289	290	291	292	293	294
295	296	297	298	299	300
301	302	303	304	305	306
307	308	309	310	311	312
313	314	315	316	317	318
319	320	321	322	323	324
325	326	327	328	329	330
331	332	333	334	335	336
337	338	339	340	341	342
343	344	345	346	347	348
349	350	351	352	353	354
355	356	357	358	359	360
361	362	363	364	365	366
367	368	369	370	371	372
373	374	375	376	377	378
379	380	381	382	383	384
385	386	387	388	389	390
391	392	393	394	395	396
397	398	399	400	401	402
403	404	405	406	407	408
409	410	411	412	413	414
415	416	417	418	419	420
421	422	423	424	425	426
427	428	429	430	431	432
433	434	435	436	437	438
439	440	441	442	443	444
445	446	447	448	449	450
451	452	453	454	455	456
457	458	459	460	461	462
463	464	465	466	467	468
469	470	471	472	473	474
475	476	477	478	479	480
481	482	483	484	485	486
487	488	489	490	491	492
493	494	495	496	497	498
499	500	501	502	503	504
505	506	507	508	509	510
511	512	513	514	515	516
517	518	519	520	521	522
523	524	525	526	527	528
529	530	531	532	533	534
535	536	537	538	539	540
541	542	543	544	545	546
547	548	549	550	551	552
553	554	555	556	557	558
559	560	561	562	563	564
565	566	567	568	569	570
571	572	573	574	575	576
577	578	579	580	581	582
583	584	585	586	587	588
589	590	591	592	593	594
595	596	597	598	599	600
601	602	603	604	605	606
607	608	609	610	611	612
613	614	615	616	617	618
619	620	621	622	623	624
625	626	627	628	629	630
631	632	633	634	635	636
637	638	639	640	641	642
643	644	645	646	647	648
649	650	651	652	653	654
655	656	657	658	659	660
661	662	663	664	665	666
667	668	669	670	671	672
673	674	675	676	677	678
679	680	681	682	683	684
685	686	687	688	689	690
691	692	693	694	695	696
697	698	699	700	701	702
703	704	705	706	707	708
709	710	711	712	713	714
715	716	717	718	719	720
721	722	723	724	725	726
727	728	729	730	731	732
733	734	735	736	737	738
739	740	741	742	743	744
745	746	747	748	749	750
751	752	753	754	755	756
757	758	759	760	761	762
763	764	765	766	767	768
769	770	771	772	773	774
775	776	777	778	779	780
781	782	783	784	785	786
787	788	789	790	791	792
793	794	795	796	797	798
799	800	801	802	803	804
805	806	807	808	809	810
811	812	813	814	815	816
817	818	819	820	821	822
823	824	825	826	827	828
829	830	831	832	833	834
835	836	837	838	839	840
841	842	843	844	845	846
847	848	849	850	851	852
853	854	855	856	857	858
859	860	861	862	863	864
865	866	867	868	869	870
871	872	873	874	875	876
877	878	879	880	881	882
883	884	885	886	887	888
889	890	891	892	893	894
895	896	897	898	899	900
901	902	903	904	905	906
907	908	909	910	911	912
913	914	915	916	917	918
919	920	921	922	923	924
925	926	927	928	929	930
931	932	933	934	935	936
937	938	939	940	941	942
943	944	945	946	947	948
949	950	951	952	953	954
955	956	957	958	959	960
961	962	963	964	965	966
967	968	969	970	971	972
973	974	975	976	977	978
979	980	981	982	983	984
985	986	987	988	989	990
991	992	993	994	995	996
997	998	999	1000	1001	1002

I.	II.	III.	IV.	V.	VI.	VII.
16.2913	1.10381	0.88029	2237	537	488	49
21.2913	1.718048	0.984883	2503	266	248	16
26.2913	imaginary				94	94
31.2913	"				26	26
36.2913	"				2	2
41.2913	"				1	1
46.2913	"			
51.2913	"				1	1

This function fits the observed distribution better than the normal frequency function or the transformation discussed in section II for the portion of the range which gives real values. For the positive end of the range and the first four frequency groups of the negative range, this function describes the observed distribution fairly well. From the fourth term on, the terms on the negative side are imaginary. Professor Edgeworth states* "that the difficulty from occurrence of imaginary values is not apprehended". In this example imaginary values of α and κ do not occur, but there is difficulty from the values of ε becoming imaginary.

Here another problem represents itself, namely within what interval must ε remain in order that all negative values of ε will remain in the negative region for the transformed function X .

$$X = 19.1028\varepsilon + 3.8275\varepsilon^2$$

but $\varepsilon = -t$ where t is positive. When is $-19.1028t + 3.8275t^2 \leq 0$?

$$3.8275t^2 < 19.1028t ,$$

$$t < 5.$$

Therefore ε must lie within the interval $0 < \varepsilon < -5$ so that any negative value of ε will give a negative X .

* International Congress of Mathematicians, 1912, vol. II, p. 461.

Pearson's criterion of best fit* applied for real values of ε gives $\chi^2=44.5$ and the probability $p=0.0053$ that deviations as great as or greater than these would occur under random sampling, where

$$\chi^2 = \frac{\left(\frac{\text{squares of differences of theoretical and observed frequencies}}{\text{(theoretical frequencies)}} \right)}$$

Another transformation which might be of interest for some special cases is to replace the variable x by X where $X=x+x^2$, $(1+2x)dx=dX$, $dx=$

$$\frac{dX}{1+(-1\pm\sqrt{1+4X})} \cdot \frac{-(x-a)^2}{2\sigma^2} dx = \int \frac{1}{-1\pm\sqrt{1+4X}} \cdot e^{-\frac{[(-1\pm\sqrt{1+4X})-a]^2}{2\sigma^2}} dX$$

It is necessary that the median of the normal curve should be sufficiently far above the origin so that all variates be positive.

Replacing each x by X and solving for the constants by method of moments we get

$$\frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty X^n e^{-\frac{[(1+\sqrt{1+4X})-a]^2}{2\sigma^2}} \cdot dX = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x+x^2)^n e^{-\frac{(x-a)^2}{2\sigma^2}} dx.$$

First moment.

$$\begin{aligned} \xi &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x+x^2) e^{-\frac{(x-a)^2}{2\sigma^2}} dx, \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} [(x'+a) + (x'+a)^2] e^{-\frac{x'^2}{2\sigma^2}} dx, \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} [(x'+a+\frac{1}{2})^2 - \frac{1}{4}] e^{-\frac{x'^2}{2\sigma^2}} dx, \\ \xi &= (a + a^2) + \sigma^2. \quad \text{I.} \end{aligned}$$

Second moment.

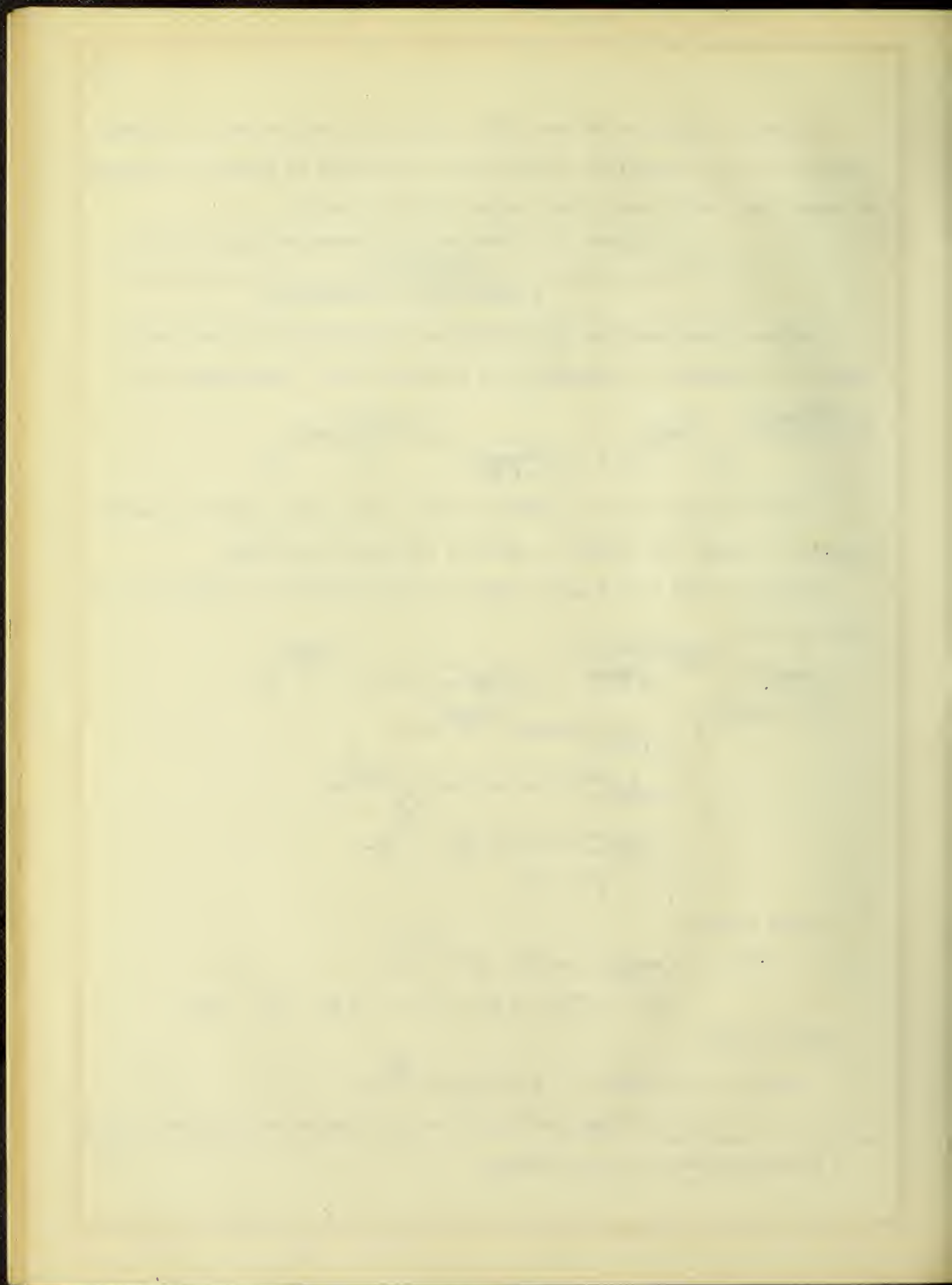
$$\begin{aligned} \xi^2 + \mu_2 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} [(x+a+\frac{1}{2})^2 - \frac{1}{4}]^2 e^{-\frac{x^2}{2\sigma^2}} dx \\ &= 3\sigma^4 + \sigma^2(6a^2 + 6a + 1) + a^4 + 2a^3 + a^2 \quad \text{II.} \end{aligned}$$

Third moment.

$$\begin{aligned} \xi^3 + 3\xi\mu_2 + \mu_3 &= \int_{-\infty}^{+\infty} [(x + a + \frac{1}{2})^2 - \frac{1}{4}]^3 e^{-\frac{x^2}{2\sigma^2}} dx \\ &= 15\sigma^6 + 3\sigma^4(15a^2 + 15a + 3) + \sigma^2(15a^4 + 30a^3 + 18a^2 + 3a) + a^6 + a^5 + 3a^4 + a^3. \end{aligned}$$

* Biometrika, vol. I, pp. 155-161.

III.



Substituting the value of g from equation I into equation (2) and (3)

$$\mu_2 = 2\sigma^4 + \sigma^2(4a^2 + 4a + 1) \quad (4)$$

$$\begin{aligned} \mu_3 &= 11\sigma^6 + (9a^2 + 9a + 2) \cdot 3\sigma^4 \quad (5) \\ &= 11\sigma^6 + 2(4a^2 + 4a + 1) \cdot 3\sigma^4 + (a^2 + a) \cdot 3\sigma^4. \end{aligned}$$

From equation (4)

$$4a^2 + 4a + 1 = \frac{\mu_2 - 2\sigma^4}{\sigma^2} \quad (6)$$

Substituting the value of the left hand member of (6) in (5)

$$\mu_3 = 11\sigma^6 + 6\sigma^4 \left[\frac{\mu_2 - 2\sigma^4}{\sigma^2} \right] + \left[\frac{\mu_2 - 2\sigma^4}{4\sigma^2} \right] \cdot 3\sigma^4 - \frac{3}{4}\sigma^4,$$

$$4\mu_3 + 10\sigma^6 + 3\sigma^4 - 27\sigma^2\mu_2 = 0$$

Putting $\sigma^2 = f$,

$$10f^3 + 3f^2 - 27f\mu_2 + 4\mu_3 = 0,$$

$$f^3 + 0.3f^2 - 2.7f\mu_2 + .4\mu_3 = 0.$$

Since the equation in a is one of the second degree the solution of the two equations is facilitated by substituting the value of $4a^2+4a+1$ from equation (6) into equation (5).

$$V. \text{ TRANSFORMATION } X = a_1\xi + a_2\xi^2 + a_3\xi^3.$$

This type of transformation is more general, perhaps, and will change the general form of the curve. While ξ^2 tends to pile the variates near the origin and in the negative region ξ^3 has a more flattening and distributing effect.

If $y=f(X)$, then where $f(\xi)$ is the normal function with the standard deviation $\sigma=\frac{1}{\sqrt{2}}$ or $\sigma\sqrt{2}=1$ and $X=a(\xi+\kappa\xi^2+\lambda\xi^3)$, then the n^{th} moment about the median is

$$\int_{-\infty}^{\infty} X^n f(X) dX = \int_{-\infty}^{\infty} a^n (\xi+\kappa\xi^2+\lambda\xi^3)^n e^{-\xi^2} d\xi.$$

The required constants a , κ and λ can be solved for by method of moments.

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The moments are calculated from the median. For convenience we shall use Edgeworth's notation*. M_1, M_2 , etc., represent the moments about the median.

$$M_1 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} a(\xi + \kappa \xi^2 + \lambda \xi^3) e^{-\xi^2} d\xi \\ = \frac{1}{2} a \kappa.$$

$$M_2 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} a^2 (\xi + \kappa \xi^2 + \lambda \xi^3)^2 e^{-\xi^2} d\xi \\ = a^2 \left(\frac{1}{2} + \frac{3}{4} \kappa^2 + \frac{3}{2} \lambda + \frac{15}{8} \lambda^2 \right).$$

$$M_3 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} a^3 (\xi + \kappa \xi^2 + \lambda \xi^3)^3 e^{-\xi^2} d\xi \\ = a^3 \left(\frac{9}{4} \kappa^2 + \frac{15}{8} \kappa^3 + \frac{45}{4} \kappa \lambda + \frac{315}{16} \kappa \lambda^2 \right)$$

$$\mu_1 = M_1 = \frac{1}{2} a \kappa. \quad (1)$$

$$\mu_2 = M_2 - M_1^2 = \frac{a^2}{2} \left(1 + \kappa^2 + 3\lambda + \frac{15}{4} \lambda^2 \right) \quad (2)$$

$$\mu_3 = M_3 - 3M_1\mu_2 - M_1^3 = a^3 \kappa \left(\frac{3}{2} + \kappa^2 + 9\lambda + \frac{135}{8} \lambda^2 \right) \quad (3)$$

$$\text{Since } \beta_1 = \frac{\mu_3}{\mu_2} \text{ and } \eta^{**} = \frac{\mu_4}{\mu_2^2} - 3 \quad (4)$$

we may solve for the constants a, κ, λ .

$$\beta_1 = \frac{\mu_3}{\mu_2} = \frac{8\kappa^2 \left(\frac{3}{2} + \kappa^2 + 9\lambda + \frac{135}{8} \lambda^2 \right)^2}{\left(1 + \kappa^2 + 3\lambda + \frac{15}{4} \lambda^2 \right)^3}.$$

$$\eta = \frac{\mu_4}{\mu_2^2} - 3 = \frac{4(6\kappa^2 + 3\lambda + 3\kappa^4 + 54\kappa^2\lambda + 27\lambda^2 + 135\kappa^2\lambda^2 + \frac{405}{4}\lambda^3 + \frac{1215}{8}\lambda^4)}{(1 + \kappa^2 + 3\lambda + \frac{15}{4}\lambda^2)^2}$$

We may use $\chi = \kappa^2$,

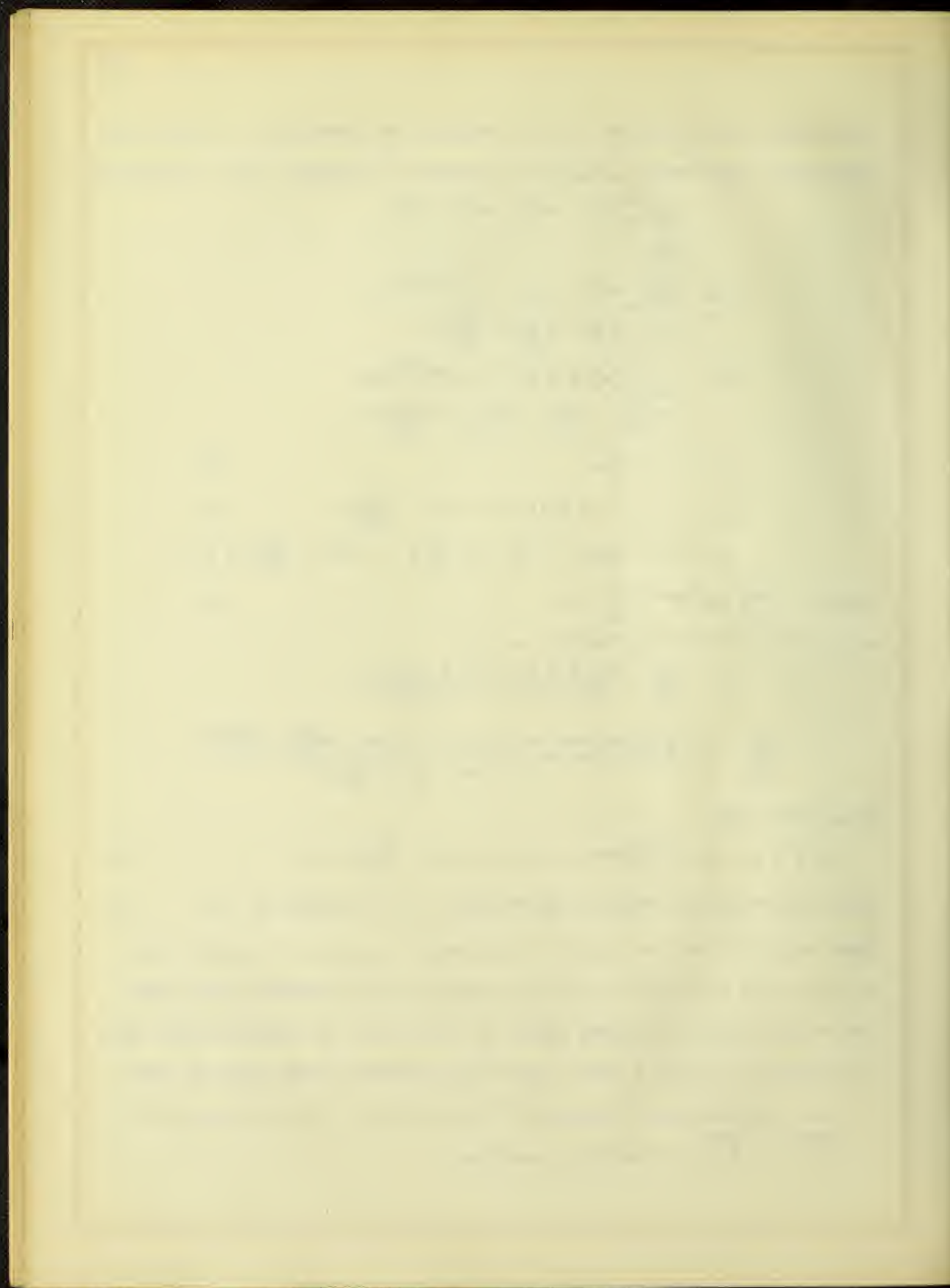
$$8\chi \left(\frac{3}{2} + \chi + 9\lambda + \frac{135}{8} \lambda^2 \right)^2 - \beta_1 \left(1 + \chi + 3\lambda + \frac{15}{4} \lambda^2 \right)^3 = 0 \quad (5)$$

$$4(6\chi + 3\lambda + 3\chi^2 + 54\chi\lambda + 27\lambda^2 + 135\chi\lambda^2 + \frac{405}{4}\lambda^3 + \frac{1215}{4}\lambda^4) - \eta \left(1 + \chi + 3\lambda + \frac{15}{4} \lambda^2 \right)^2 = 0 \quad (6)$$

These equations are not always so difficult to solve as they seem. This is due to the fact that κ and λ are usually small fractions. The higher powers will, therefore, become small. In trying out the numerical data used in sections II and IV, I made a first approximation, using only the first

* Fifth International Congress of Mathematicians, Proceedings, vol. II, 1912, p. 430.

** The $\eta = \beta_2 - 3$ in Pearson's notation.



powers of κ and λ . Then from equations (5) and (6) we get:

$$18\chi - 0.707005(1 + 3\chi + 9\lambda) = 0 \quad (7)$$

and $24\chi + 12\lambda - 1.833691(1 + 3\chi + 6\lambda) = 0 \quad (8).$

Solving for χ and λ

$$\lambda = 0.1015,$$

$$\chi = 0.0852.$$

Here λ and χ seem too large, so a second approximation was made. This was done by substituting for each λ and χ in equations (5) and (6) the value of λ and χ plus some new λ and χ say λ' and χ' . Thus $\lambda = 0.1015 + \lambda'$, and $\chi = 0.0852 + \chi'$. Only the first powers of χ' and λ' were retained because of the difficulty of solution of higher powers.

$$\chi' = -0.030603,$$

$$\lambda' = -0.037525.$$

Then

$$\chi = \chi' + 0.08529$$

$$\chi = 0.05469$$

$$\lambda = \lambda' + 0.101529$$

$$\lambda = 0.07303.$$

Since $\chi = \kappa^2$, $\kappa = \pm 0.233679.$

These values of κ and λ are substituted in equation (2) and a is solved for.

$$a^2 = 293.3694$$

$$a = \pm 17.1425.$$

These constants have been determined in a more precise way than in section IV. Because of the immense amount of labor that would be involved in solving a cubic equation by Horner's method for each value of ξ - and in choosing the root appropriate for the transformation and because κ, λ, \dots

<p>THE</p> <p>REPORT</p> <p>OF THE</p> <p>COMMISSIONERS OF THE</p> <p>LAND OFFICE</p> <p>FOR THE YEAR</p> <p>1864</p>	
<p>IN</p> <p>THE</p> <p>UNITED STATES</p> <p>OF AMERICA</p>	
<p>WASHINGTON:</p> <p>1865</p>	
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<p>AND</p> <p>FOR</p> <p>SALE</p> <p>BY</p> <p>THE</p> <p>COMMISSIONERS</p> <p>OF THE</p> <p>LAND OFFICE</p>	
<p>WASHINGTON</p>	

are a decreasing sequence, that is λ is smaller than κ , it was thought best to retain only the first and second powers of ξ .

From
$$X = a(\xi + \kappa \xi^2)$$

$$\xi = \frac{-1 \pm \sqrt{1 + \frac{4\kappa X}{a}}}{2\kappa}.$$

In table IV the theoretical frequency of this sort of transformation is given. Column I gives the value of X as calculated from the median; col. II gives ξ calculated from a , κ , X where $\xi = \frac{-1 \pm \sqrt{1 + \frac{4\kappa X}{a}}}{2\kappa}$; col. III, $\varphi(\xi)$; col. IV, $\varphi(\xi) \times \frac{5082}{2}$; col. V the theoretical frequency; col. VI the observed frequency; col. VII the residuals.

TABLE IV.

Above the median.

I.	II.	III.	IV.	V.	VI.	VII.
3.7087	0.20637	0.22956	583	831	817	14
8.7087	0.45898	0.48366	1229	646	600	46
13.7087	0.68844	0.66970	1702	473	513	40
18.7087	0.902113	0.79745	2026	324	315	9
23.7087	1.09965	0.88009	2236	210	208	2
28.7087	1.28715	0.93133	2367	131	108	23
33.7087	1.46496	0.961698	2444	77	75	2
38.7087	1.63496	0.979232	2488	44	38	6
43.7087	1.79579	0.989079	2513	25	28	3
48.7087	1.95321	0.9942523	2526	13	18	5
53.7087	2.101108	0.9970339	2533	7	17	10
58.7087	2.24568	0.998606	2537	4	5	1
63.7087	2.38564	0.9992582	2539	2	4	2
68.7087	2.52658	0.9996473	2540	1	2	1
73.7087	2.65404	0.999815	2541	1	1	0
78.7087	2.907407	0.99993	2541	0	2	2
83.7087	3.02934	0.9999809	2541	0	0	0



I.	II.	III.	IV.	V.	VI.	VII.
88.7087	3.14875	0.9999915	2541	0	0	0
93.7087	3.26559	0.999998	2541	0	1	0

Below the median.

1.2913	0.076707	0.097577	248			
6.2913	0.401854	0.43016	1093	345	746	99
11.2913	0.81323	0.74992	1905	312	724	88
16.2913	1.424588	0.956059	2429	524	488	36
21.2913	imaginary				248	248
26.2913	"				94	94
31.2913	"				26	26
36.2913	"				2	2
41.2913	"				1	1
46.2913	"				0	0
51.2913	"				1	1

The upper half of this transformed curve describes the given distribution very well. There is, however, absolutely no fit below the centroid, the values becoming imaginary after the third frequency group below the origin. It is to be observed that the negative values appear in this transformation sooner than in the other transformations. This is due to the fact that α is not large enough as compared with κ .

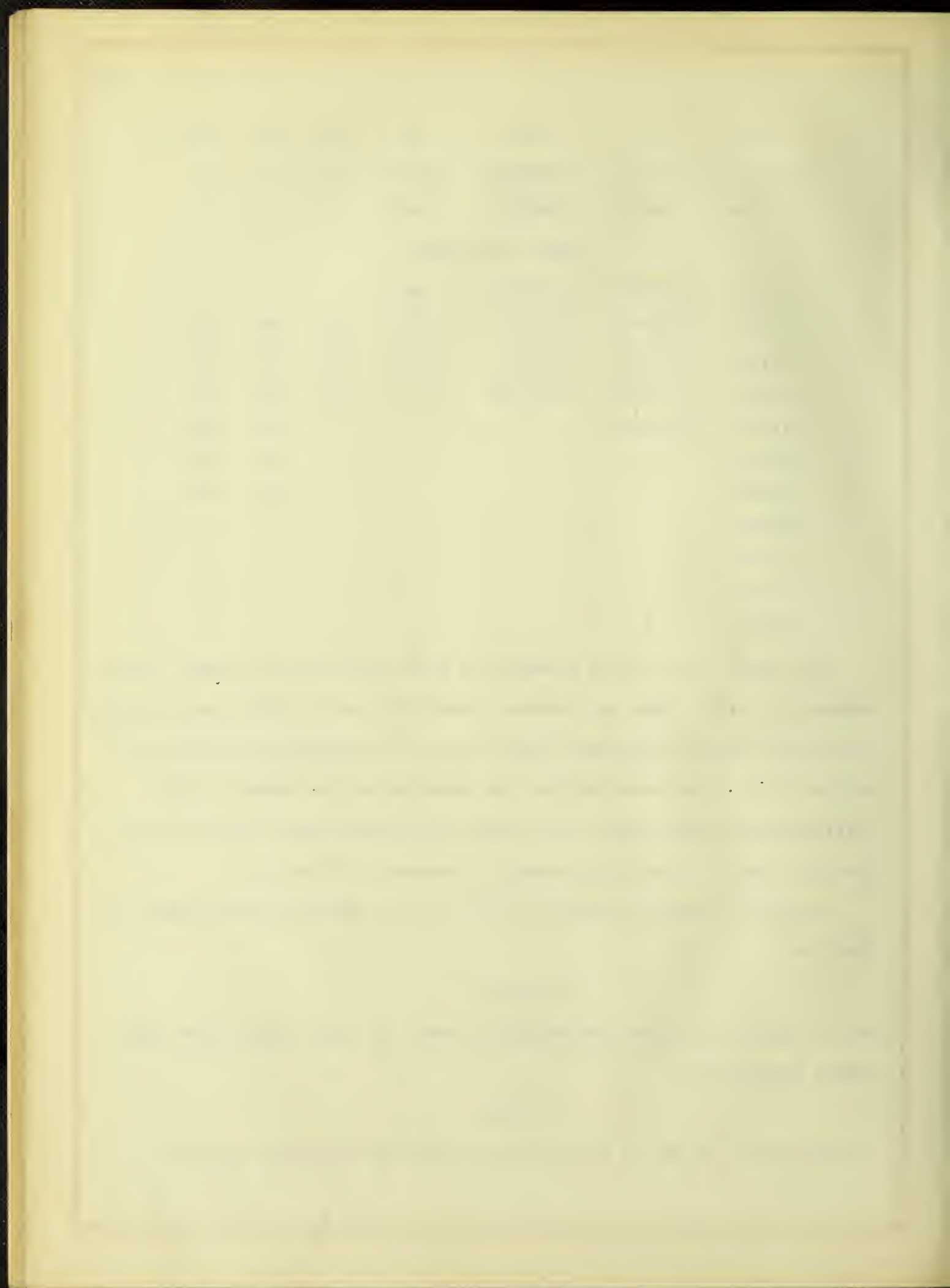
Applying Pearson's criterion of fit to the part above the median, we find that

$$\chi^2 = 22.47$$

and the probability that deviations as great as these would occur under random sampling is

$$P = 0.055 .$$

In conclusion, it may be fitting to say that an indefinite number of



transformations are possible. The difficulty is in selecting, a priori, the appropriate transformation for the particular data given. The transformed functions considered in this paper have advantages in being capable of application to data which in general appearance deviates from the normal curve. This fact is shown by the graphs of the transformed function (figs. 1-8).

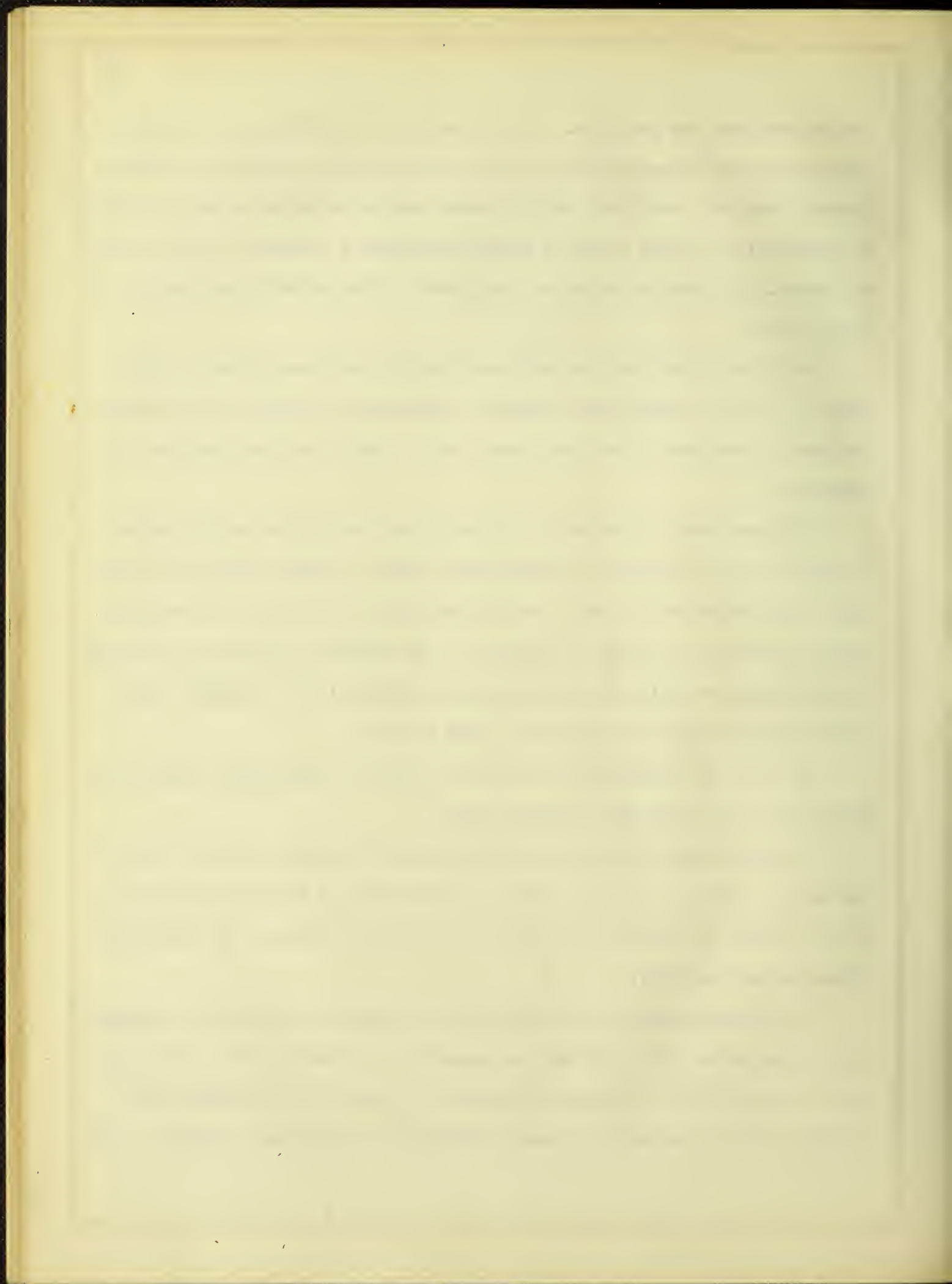
The normal function does not describe the data considered in this paper. While the transformed function discussed in section II is better for part of the data, there are other parts which this function does not describe.

The Pearsonian criterion of fit could not be applied as the tables do not extend far enough, in other words there is little probability that such great deviation is due to random sampling. If the part of the theoretical distribution above the median is considered by itself the probability that such deviation is due to random sampling is $P = 0.000001$. Thus even the part above the median is a very bad fit.

The function considered in section IV gives a much better description except where the imaginary numbers appear.

The theoretical distribution of section V describes the part above the median very well, as was found by application of Pearson's criterion of fit. Below the median there are only three real values, the remaining values being imaginary.

The disappearance of the difficulty of imaginary values here obtained is to be expected when κ is smaller compared to α than it is in this problem. This condition is doubtless realized for cases which deviate but slightly from the normal. It seems difficult to determine, a priori, other



conditions under which $\kappa \mathcal{E}^2$ is sufficiently small so that we entirely avoid the occurrence of imaginary values of \mathcal{E} to correspond to numerically large negative values of X .

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FIGURE No-1.

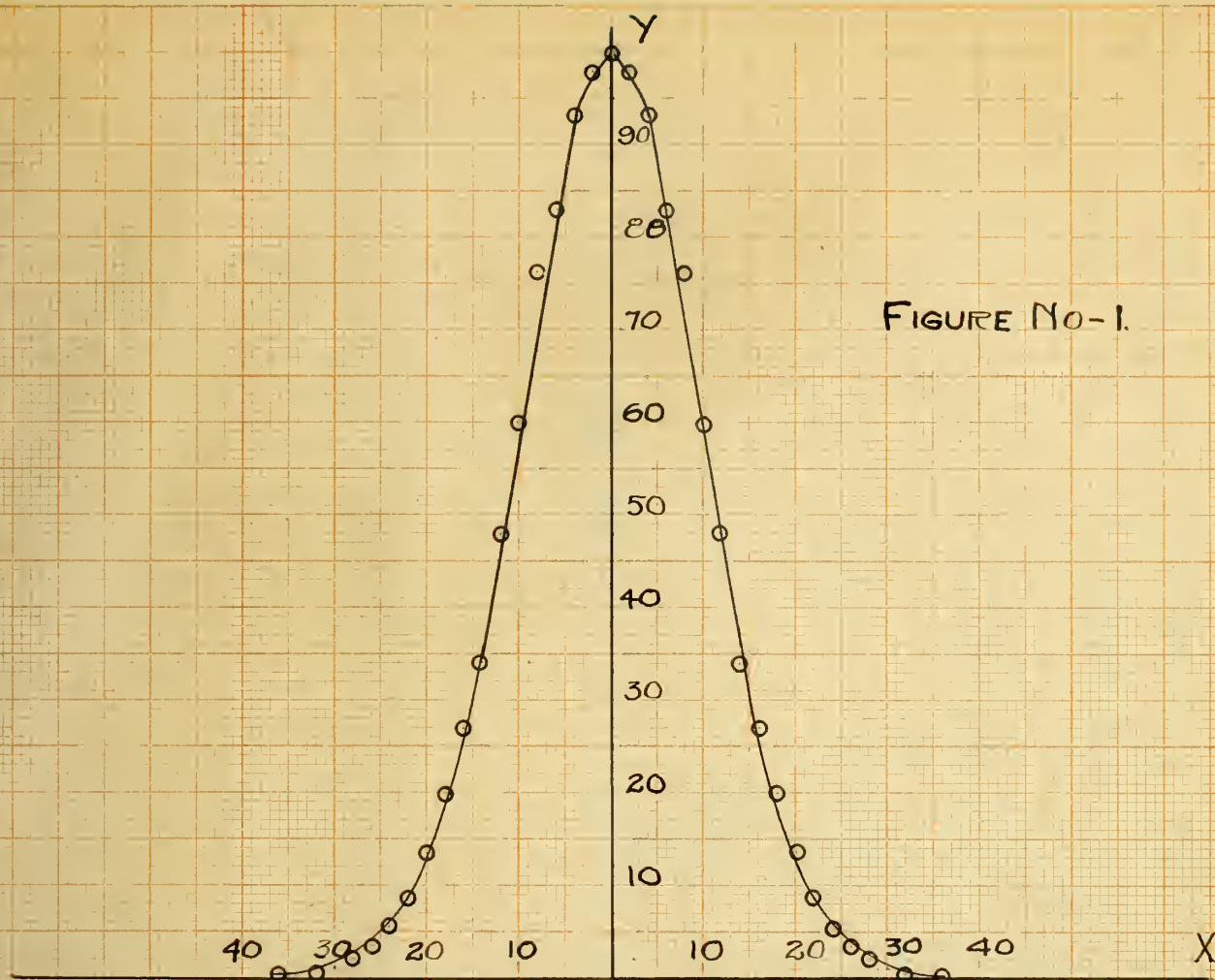
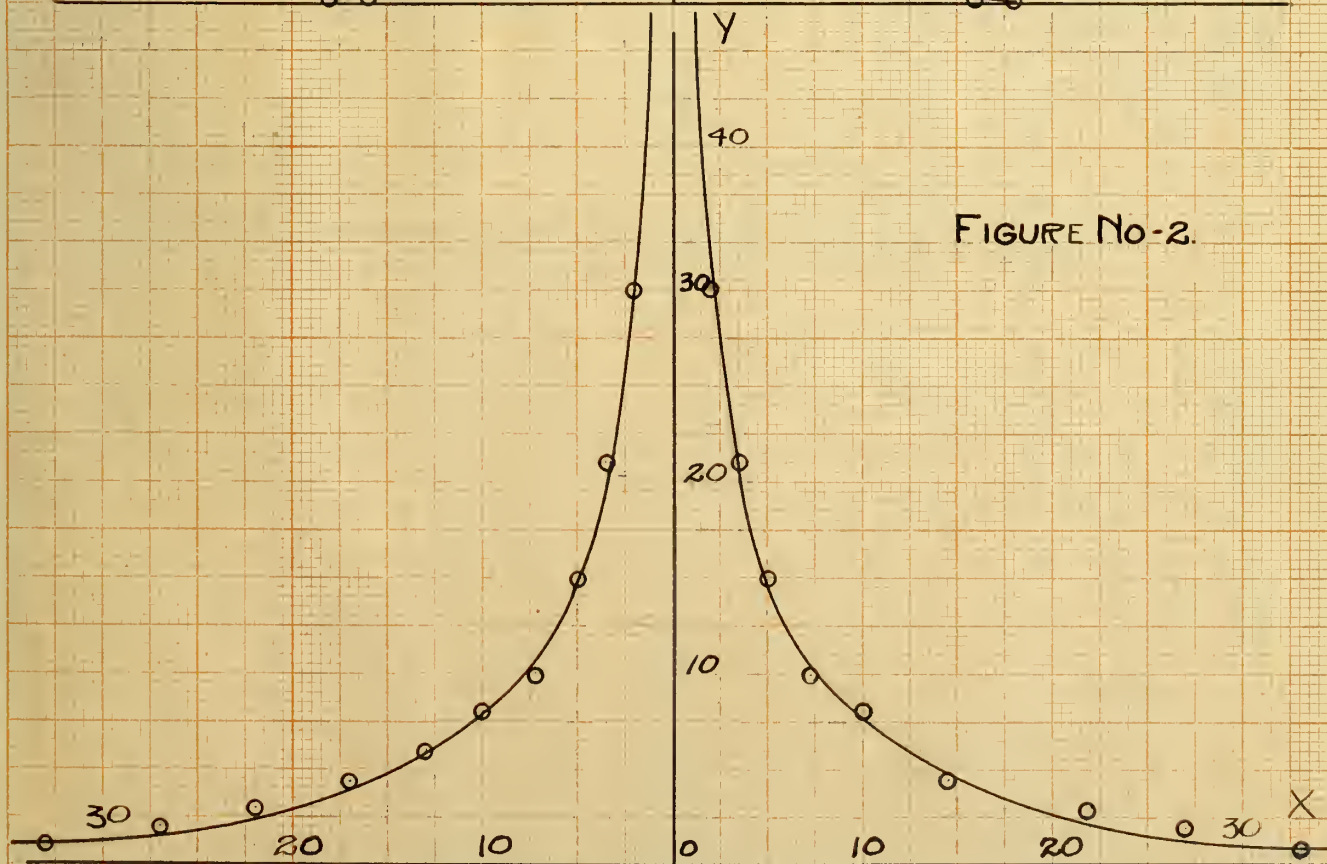
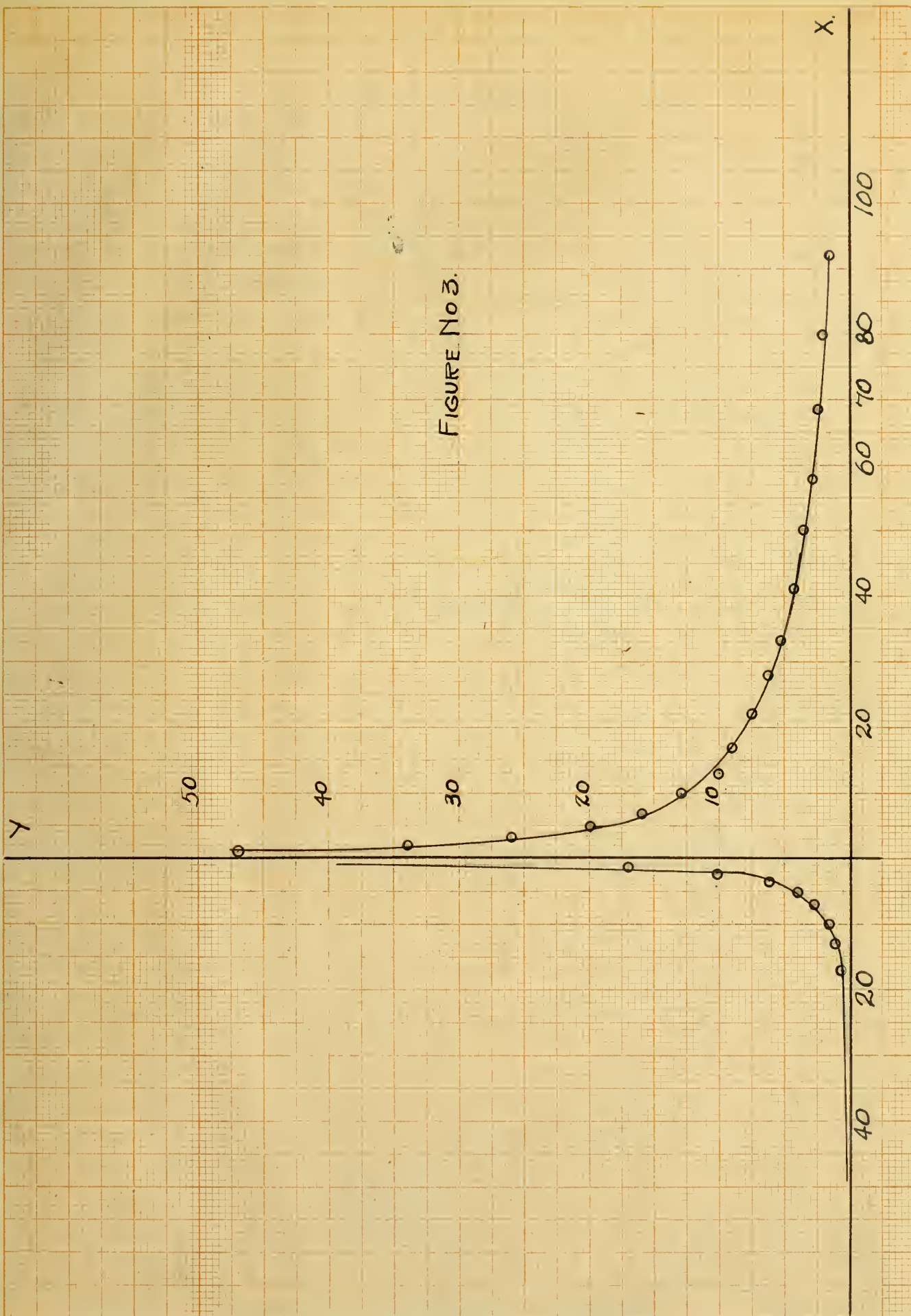


FIGURE No-2.



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FIGURE No 3.



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FIGURE No-4

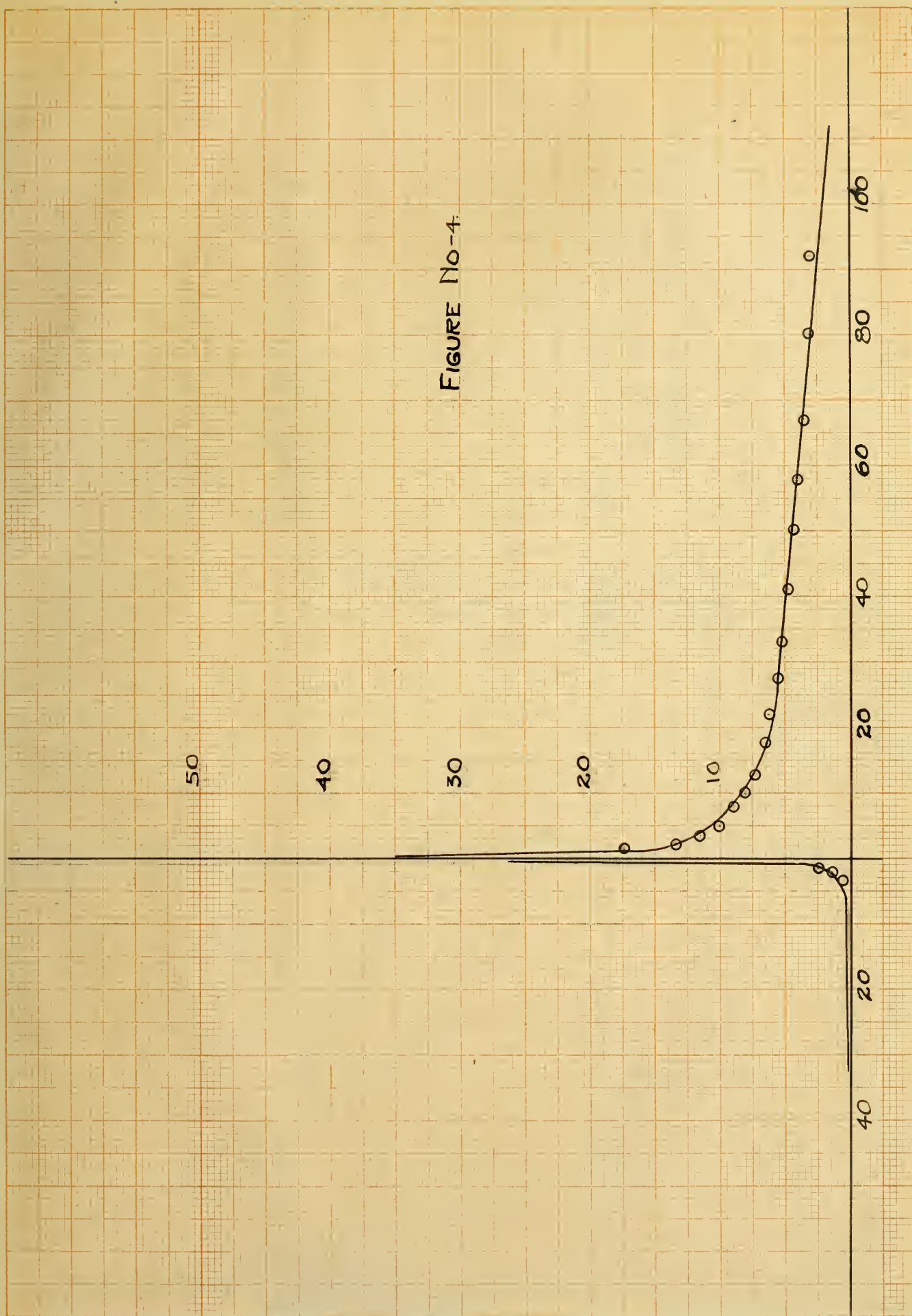


FIGURE NO-5 FULL LINE

FIGURE NO-6 DOTTED LINE

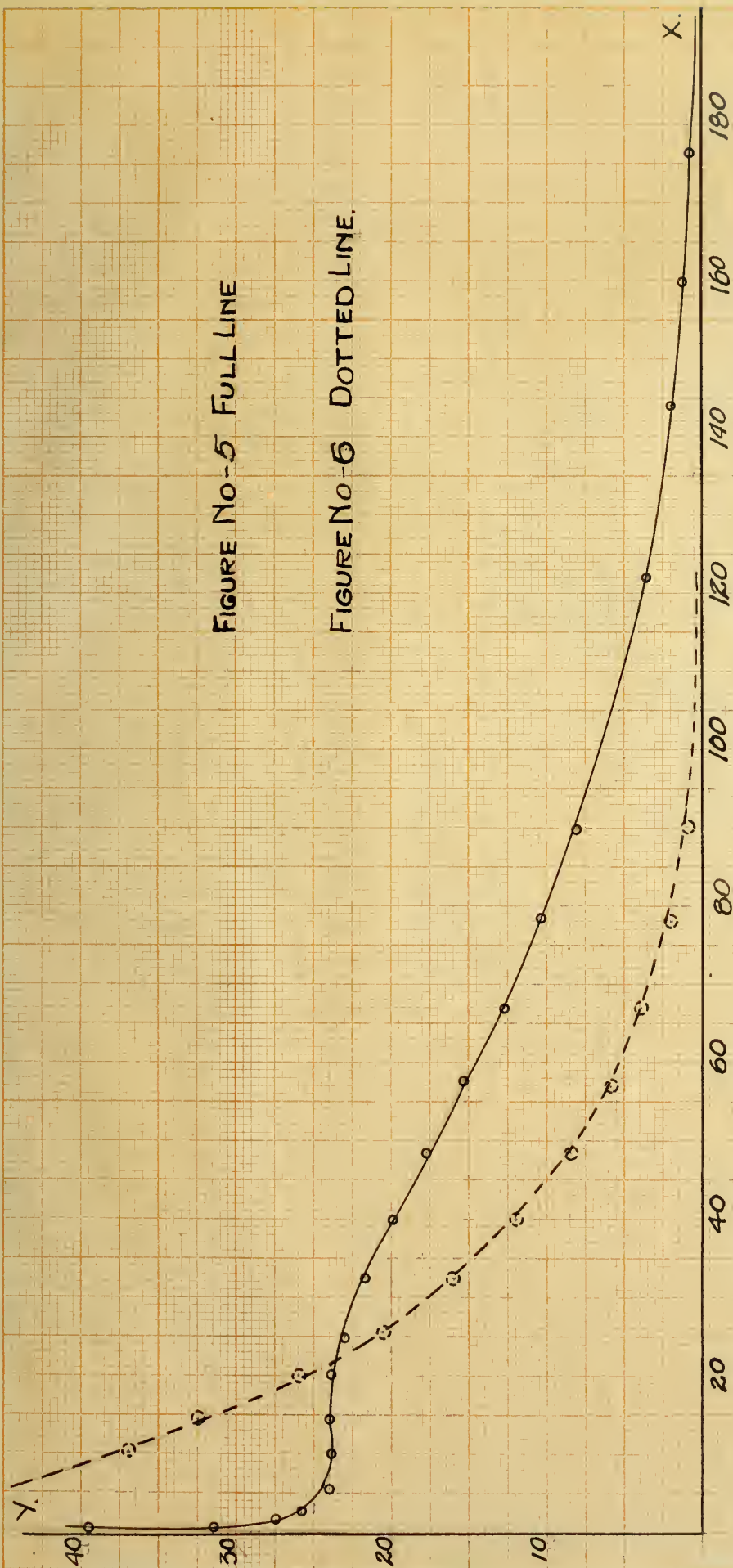
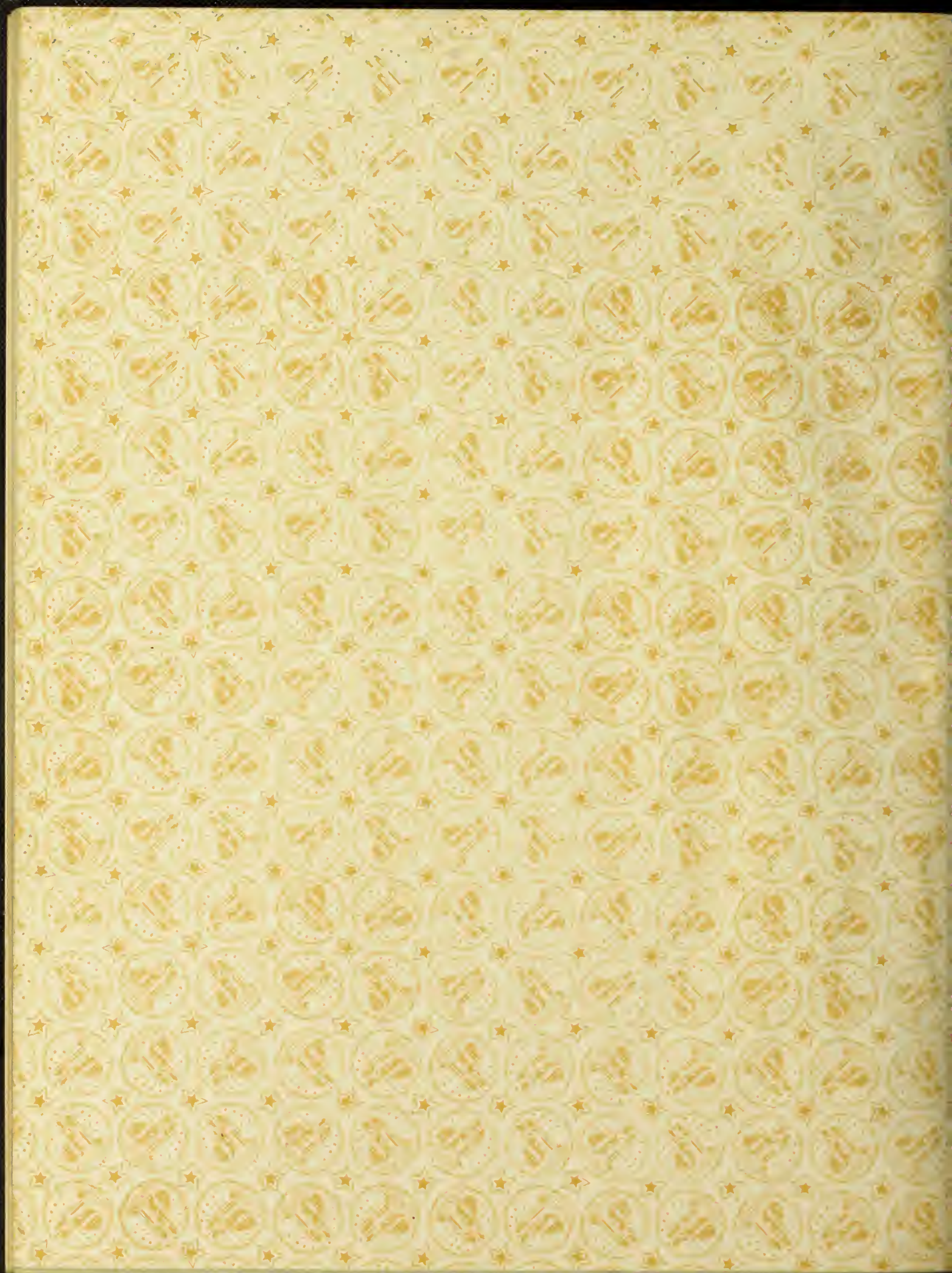


FIGURE 7 HEAVY LINE.
FIGURE 8 DOTTED LINE.





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